

# Optimal Display of Iceberg Orders\*

Gökhan Cebiroğlu

Ulrich Horst

Humboldt-Universität zu Berlin

Humboldt-Universität zu Berlin

Department of Mathematics

Department of Mathematics

Unter den Linden 6

Unter den Linden 6

10099 Berlin

10099 Berlin

goekhan.cebiroglu@db.com

horst@math.hu-berlin.de

August 23, 2011

## Abstract

We develop a sequential trade model of Iceberg order execution in a limit order book. The Iceberg-trader has the freedom to expose his trading intentions or (partially) shield the true order size against other market participants. Order exposure can cause drastic market reactions (“market impact”) in the end leading to higher transaction costs. On the other hand the Iceberg trader faces a loss-in-priority when he hides his intentions, as most electronic limit order books penalize the usage of hidden liquidity. Thus the Iceberg-trader is faced with the problem to find the right trade-off. Our model provides optimal exposure strategies for Iceberg traders in limit order book markets. In particular, we provide a range of analytical statements that are in line with recent empirical findings on the determinants of trader’s exposure strategies. In this framework, we also study the market impact also market impact of limit orders. We provide optimal exposure profiles for a range of high-tech stocks from the US S&P500 and how they scale with the state-of-the-book. We finally test the Iceberg’s performance against the limit orders and find that Iceberg orders can significantly enhance trade performance by up to 60%.

**JEL classification:** C51,C60,C67,D01,D4,D49,G1

**Keywords:** Hidden Liquidity, Iceberg Orders, Limit Order Book, Market Impact of Limit Orders, Optimal Exposure, Trading Strategies, Iceberg versus Limit Order, Pre-trade transparency, Agency-Trading.

---

\*We thank Daniel Nehren, Mark DiBattista, Boris Drovetsky, Nikolas Hautsch and Ruihong Huang for support and valuable comments and suggestions. We also thank participants of the 6th World Congress of the Bachelier Finance Society (2010), the Princeton-Humboldt Finance Workshop (2009), the Fields Quantitative Finance Seminar (2009) and the Conference on Modeling and Managing Financial Risks (2011) for helpful discussions and comments. Financial support and data provision from Deutsche Bank is gratefully acknowledged. Horst gratefully acknowledges financial support through the SFB 649 *Economic Risk*.

# 1 Introduction

In almost all markets, trade-initiation requires the ex-ante commitment of one party at least, that is, at least one trader needs to expose his intention to trade. In fact, *exposure* of trading intentions lies at the very basis of trading itself. In order-driven markets, for instance, liquidity suppliers expose limit orders to attract liquidity demanders. In view of the inherent information-leakage that is associated with it, exposure can generate risks as well though. Copeland and Galai (1983) argue that limit orders give away *free options* to *better-informed* market participants and that limit order-owners face the risk of getting *picked-off*. Harris (1996, 1997, 2003) explains how the presence of so called *parasitic traders* can increase trading costs when exposing limit orders: at the expense of the originator, parasitic traders exploit the *options value* out of the standing limit orders, by employing so called *quote-matching* strategies.<sup>1</sup> Bessembinder, Panayides, and Venkatamaran (2009) empirically confirm, that order exposure in limit order book markets can increase, while hiding one's trading interests can substantially minimize investors' transaction costs. Consequently, markets have understood that providing investors with means to control and mitigate exposure-related risks might attract even more liquidity to their trading platforms. Therefore, more and more securities markets are providing *hidden liquidity* as a strategic trade-tool to investors, either by virtue of "dark" exchanges with limited/zero pre-trade transparency like *Dark Pools* or *Crossing Networks*, or by introducing *hidden* or so called *Iceberg Orders* on traditional exchanges. The latter enables traders to control information disclosure of their standing orders by hiding the true order size.<sup>2</sup> Showing only a fraction of their overall size, *Icebergs* automatically slice the order into a sequence of smaller orders and submit them in a timely succession. Doing this manually, would generally require a highly sophisticated *Order and Execution Management System* (OMS/EMS). *Icebergs* are thus convenient trade tools, particularly for less sophisticated large traders that frequently/occasionally face substantial exposure-risk.

A growing body of empirical studies indicate the wide-spread use of *hidden* orders. For instance, Pascual Gasco and Veredas (2008) report that 26% of all trades on the Spanish Stock Exchanges involve hidden volume. Frey and Sandas (2009) report that 9.3% of *submitted* and 15.9% of *executed* shares contain Iceberg orders on the German Xetra Stock Exchange. De Winne and D'Hondt (2004; 2007; 2009) report that 27.2% (20.4) of the total liquidity in the book is hidden for the French CAC40 (Belgian BEL20) exchanges and moreover that the *hidden ratios* can even reach 50% at the best limit prices. Tuttle (2003) finds that around 25% of liquidity for all NASDAQ National Market quotes are hidden. Further studies confirm that hidden liquidity is particularly prevalent among large investors: D'Hondt et al. (2004) report that 81% of orders with total sizes in the largest quartile are Icebergs or (partly) hidden orders. Supplementing this findings, Frey and Sandas (2009) find that Iceberg orders are on average 12-20 times larger than limit orders. Despite the many benefits for liquidity-providing investors, markets need to balance investors' demand for hidden liquidity with their natural desire to trade at transparent

---

<sup>1</sup>Consistent with this reasoning Aitken et al. (2001) and Bessembinder et al. (2009) report that hidden orders are used to reduce the *options value*. The findings of Harris (1996) and De Winne and D'Hondt (2009) suggest that traders are willing to expose more when the minimum tick size is large, i.e. when the risk of getting front-run is low.

<sup>2</sup>In this sense, *Iceberg* is an allusion to the arctic Iceberg, where only a fraction, namely its top, is openly displayed.

(“fair”) prices and in particular to understand the price formation process. Therefore, markets put limits on *pre-trade opacity*; the majority of today’s electronic exchanges give visible liquidity higher *priority-in-execution* over hidden liquidity.<sup>3</sup> The beneficial reduction in exposure-risk comes at the cost of a loss in execution-priority, i.e. an increased execution risk. Traders thus need to balance these two antagonistic sources of risk. This paper proposes a simple microstructure model for analyzing optimal display sizes that yields many empirically testable hypotheses.

## 1.1 Literature Review

Although the theoretic literature on limit order book models is rather rich, see Foucault, Kadan, and Kandel (2005), Goettler, Parlour, and Rajan (2005) or Rosu (2009) among many others,<sup>4</sup> “hidden” aspects have so far received relatively little attention in the more theoretical literature. Moinas (2010) and Buti and Rindi (2008) employ game-theoretic frameworks with informed agents, obtaining (stationary) equilibrium exposure strategies. Moinas introduces a sequential trade model where the display of a large limit order may cause other *impatient* traders to retreat from the market.<sup>5</sup> As a consequence, the limit order trader is faced with lower execution probability. Therefore, he uses hidden liquidity to obscure his intentions. Buti and Rindi extend the literature on *dynamic limit order markets* by allowing *strategic traders* the freedom to use (partially) hidden orders. They find that hidden orders are used to prevent traders from liquidity competition. In line with the regulator’s perspective, their simplified, agent-based approach allows them to address questions optimal market design and regulation, and to study different dimensions of market quality with respect to hidden orders. In both models - in order to solve for the equilibrium strategies and to reproduce the desired salient features - the set-up is necessarily limited and stylized.<sup>6</sup> In addition, the introduction of concepts like “information” or “utility-preferences” makes it generally hard to quantify and reconcile the models with available market data, precluding any practical applications from the *investor’s* point-of-view.

The literature on *optimal liquidation*, pioneered by the work of Bertsimas and Lo (1998) and Almgren and Chriss (1999) and recently extended by many authors including Almgren and Chriss (1999); Almgren (2003, 2001); Obizhaeva and Wang (2005); Alfonsi, Fruth, and Schied (2010) follows a more qualitative approach to modeling trading and liquidation problems in illiquid markets. Almgren and Chriss explains the need for this line of research: “[The game-theoretic ansatz] is helpful when we are interested in certain properties of the book, but quite restrictive when analyzing the optimal trading strategy”. In the spirit of the Almgren-Chriss approach Esser and Mönch (2007) consider a single-agent framework with a risk-neutral *liquidity*

---

<sup>3</sup>More precisely, the majority of today’s electronic stock exchanges enforce price priority first. Display-priority is enforced at second stage.

<sup>4</sup>See also Glosten (1994); Seppi (1997); Parlour (1998); Foucault (1999); Parlour and Seppi (2003). Parlour and Seppi (2008) provides an extensive survey on Limit Order Book models.

<sup>5</sup>In this framework, impatient traders are in effect market order traders.

<sup>6</sup>For instance, in Moinas (2010) order sizes and depths can only take three values. Buti and Rindi (2008) on the other hand, assume that large traders trade a fixed-size position, while small traders trade at smaller sizes than large trader’s display size.

*trader*<sup>7</sup>, who arrives pre-committed to trade and has a non-informational (“exogenous”) trade motive. The trader is not concerned with informational aspects and is thus blind towards *adverse selection risks*. Moreover, they assume that the trader determines his trading strategy with a view at reducing his transaction costs only. In spite of considering the action of every single agent simultaneously, the focus on the single agent’s optimal trade strategy allows to aggregate the rest agent population’s actions, by using *aggregated* order flow and price dynamics. Typically, the emerging *aggregated* processes are deemed exogenous.<sup>8</sup> Under stylized market assumptions, Esser and Mönch obtain trader’s optimal (static) Iceberg-strategy.

## 1.2 Our Contributions

Our work contributes to literature on optimal order placement in limit order books. Specifically, we consider a *liquidity* trader, with a *fixed* trading horizon using an Iceberg order. We assume that the trader has a prespecified *reference* or *benchmark price* and the additional freedom to hide a portion of the order. Order execution is governed by order arrivals and cancellations. In order to keep the analysis tractable and transparent we do not model individual order arrivals but rather introduce *aggregate* limit and market order flow volumes and cancellation ratios, that add and retract liquidity on the trader’s side of the market. The random flows determine the execution volume at terminal time; assuming a liquidation constraint, at the end of the trading period unexecuted orders are executed against the best prevailing opposite price. The price process is modeled as a separate (independent) stochastic process. The trader’s goal is to determine the optimal exposure (display size) so as to minimize his transaction costs.

In our setting, there are three main sources of risk that determine transaction costs. The first two are related to execution risk: market order traders refraining from entering the market, and increased liquidity competition at (or ahead) the submission price level.<sup>9</sup> While both scenarios of execution-risk are embedded into our model, the equilibrium-models of Moinas (2010) and Buti and Rindi (2008) only report evidence for at most one scenario. According to Moinas (2010) *impatient* traders may refrain from trading (reduced market order flow) thus increasing execution risk; according to Buti and Rindi (2008) execution risk may increase due to higher liquidity competition (increase in limit order flow). The third source of risk is introduced by the liquidation constraint, i.e., the uncertainty about the price at the end of the trading period. Our framework therefore enables us to study the effect of the execution risk, which is governed by the order flow, and adverse price movements on the optimal display decision simultaneously. The assumption that the trader’s exposure decision impacts the market dynamics, especially order flows, is central to our model and the modeling of *exposure impact*. Since the Iceberg’s exposure changes the *visible* state of the order book, in particular the quoted book-imbalance<sup>10</sup>, we assume that changes in the visible state of the order book influence the dynamics of the market. The fact that the state of the order book carries information about future market movements and that much of the explanatory power can be captured by the best quoted depth

---

<sup>7</sup>Foucault et al. (2005) use the *liquidity* trader-approach in a recent limit order book model.

<sup>8</sup>Instead of providing an additional equilibrium argument as done by Moinas (2010) and Buti and Rindi (2008), the market is taken as given. For limit order book modeling, several other studies use the exogenous-order-flow-ansatz; See Angel (1994); Domowitz and Wang (1994); Harris (1998).

<sup>9</sup>By introducing the *parasitic* trader Harris (1996) provides economical reasoning for the latter scenario.

<sup>10</sup>Book-imbalance is to be understood as the ratio between visibly quoted bid-side versus sell-side liquidity

imbalance has been extensively documented in the empirical literature; see, e.g., Bessembinder et al. (2009); Biais et al. (1995); Rinaldo (2004); Griffiths et al. (2000); Cao et al. (2009); Huang and Stoll (1994); Beber and Caglio (2005); Pascual Gasco and Veredas (2008); Chordia et al. (2002). We therefore model the *exposure impact* of our limit (Iceberg) order through its impact on the order book's imbalance.

Our work is also aimed at verifying recent findings on the *determinants of exposure decisions* by Bessembinder et al. (2009). Confirming their empirical findings, in our model the trader hides more when the order is large, *depths at the top of the book* is small, the *imbalance-excess* is on the opposite side of the book, *liquidity consumption* rates are high and/or *liquidity competition* is low. We also find that when the market's sensitivity towards the imbalance of the order book is sufficiently weak, the optimal display strategy is to expose the whole order. This is due to the fact, that in this case, exposure leads only to minor (possible) adverse market reactions, while at the same time the trader benefits to the full extent from the gain in time priority when exposing liquidity. Moreover, Bessembinder et al. report that exposure is associated with shorter *time-to-completion*, but at the same time with larger shortfall costs. We show that this is true for markets where prices drift away from the side of excess-imbalance. Due to the fact that he systematically under-estimates the *opportunity cost*, the *volume-trader* over-exposes his trading intentions to the market. We show, that there might be situations, where the opposite is true. However, our empirical observations strongly indicate, that in "realistic" markets, prices indeed drift away from the side of excess-imbalance. Our framework allows to quantify the dualism between execution risk and exposure risk. We show analytically, that the execution performance can be separated into two parts, an exposure-rewarding contribution that stems from the priority-gain and a market impact contribution that comes from the market's *reaction* towards changes in in the quoted imbalance.

Using high-frequency INET-order-message and NASDAQ quote data, we estimate the exposure impact of displayed orders, by measuring how changes in the book-imbalance affect the different components of the market: liquidity provision and liquidity consumption flow as well as the future price level. Cross-sectionally, we find that with greater exposure, prices in general move away and liquidity competition (consumption) on the same side increases (decreases). Since the imbalance is a function of the displayed order size, our results add to the research on *market impact of limit orders*.<sup>11</sup> In contrast to earlier research, by virtue of model-design, we explicitly capture the impact on flows instead of prices alone. Furthermore, our market impact estimates enable us to provide optimal display profiles for a range of high-tech stocks from the S&P 500. Particularly, we show how the optimal exposure strategy depends and scales with the state of the order book. We observe generic patterns in the optimal exposure strategies among different stocks but also report stock-specific peculiarities. Finally, we benchmark the Iceberg performance against plain limit order. Naturally, the Iceberg order always performs better than the limit order. We find that the use of Iceberg orders, instead of plain limit orders, can substantially increase the execution performance. We report performance enhancement of up to 60 %. These performance gains are particularly relevant for large orders and when the (initial) book-imbalance is skewed towards the same-side of the trade.

---

<sup>11</sup>While the impact of market orders has been extensively investigated in the optimal liquidation literature, much less effort has been devoted to the impact of limit (passive) orders. In a recent empirical work, Hautsch and Huang (2009) report that limit orders may exert significant influence on the price process in the short- and long-run.

### 1.3 Outline

The remainder of this paper is structured as follows. In Section 2, we introduce the our model, including the order flow and price dynamics and calculate the respective objective function. In Section 3, we estimate the model (market impact) parameters. We make statements about different types of market impact, presence of parasitic traders, scare-away scenario and price-retreat. In Section 4 we use the parameter estimates to calculate the Iceberg’s expected execution price for a class of order and market parameters. Furthermore, as the display size is a free parameter of the model, we are able to provide the optimal size and illustrate how the optimal display strategy scales with certain order and market parameters. Moreover, we benchmark the Iceberg performance against the case of plain limit orders. Section 5 concludes.

## 2 The Model

We consider an *Iceberg Order* trader (“she”), who trades for *liquidity* reasons. Specifically, the trader aims to buy a (large) position of  $N$  shares over a (short) trading period  $[0, T]$ . Her *reference price* is the prevailing best bid price ( $B_0$ ) at which we assume she submits an Iceberg Order.<sup>12</sup> The trader can choose to openly display any number  $\Delta \in [0, N]$  of shares in the order book. The remaining  $N - \Delta$  shares are shielded from public view and remain hidden until execution or cancellation. A random number  $V_T^\Delta$  of shares is executed before the end of the trading period. In order to enforce full liquidation at the end of the trading period, the unexecuted part  $N - V_T^\Delta$  of the order is canceled at the terminal time  $T$  and executed against standing sell limit orders at the then prevailing best ask price  $A_T^\Delta$ <sup>13</sup>. The dependence of the execution volume  $V_T^\Delta$  and best ask price  $A_T^\Delta$  on the display size  $\Delta$  accounts for the possible *impact* visible orders have on the dynamics of the order book.

**Remark 1.** *“Price aggressiveness” has been identified as a determinant of optimal display sizes in Bessembinder et al. (2009). Submitting the order at the top of the book means that aggressiveness is a not a decision variable in our model. Nevertheless, we can solve the execution problem for every single price level and than compute the optimal price level. However, this would essentially convolute our problem, without yielding further substantial insights on our central focus, namely analyzing the question of optimal exposure. This is typically a second-stage problem, once a price level has been chosen.*

The *absolute* transaction costs are given by

$$\tilde{P}^\Delta := V_T^\Delta B_0 + (N - V_T^\Delta) A_T^\Delta.$$

To facilitate performance comparisons across assets, we consider *relative* transaction costs and chose the *relative execution price* as our performance measure. We define the relative execution

---

<sup>12</sup>In *Implementation shortfall*-execution strategies, this is a fairly common set-up within the *Agency Execution*-business. The trader fixes a so-called *reference* or *decision*-price. In most cases, the best quoted prices that are available at the time of submission - i.e. either best bid or ask price - are taken to be the decision price (also frequently referred to as the “arrival price”). For further details see Johnson (2010) and Domowitz and Yegerman (2005).

<sup>13</sup>For simplicity, we assume that market orders incur no transaction costs. The impact of market orders has already been extensively studied in the recent literature. See Almgren and Chriss (1999); Alfonsi et al. (2010); Alfonsi and Schied (2010); Obizhaeva and Wang (2005).

price  $P^\Delta$  as the difference between the average trade price per share and the time-of-trade quotation normalized by the submission price  $B_0$ :

$$P^\Delta := \frac{\tilde{P}^\Delta - NB_0}{NB_0} = \frac{(N - V_T^\Delta)(A_T^\Delta - B_0)}{NB_0} = \frac{(N - V_T^\Delta)\tilde{S}_T^\Delta}{NB_0} = \left(1 - \frac{V_T^\Delta}{N}\right) S_T^\Delta. \quad (2.1)$$

The term  $(1 - \frac{V_T^\Delta}{N})$  represents the unexecuted proportion of the Iceberg order, and  $S_T^\Delta = \frac{\tilde{S}_T^\Delta}{B_0} := \frac{A_T^\Delta - B_0}{B_0}$  represents the relative difference between the benchmark and submission price  $B_0$  and the realized price  $A_T^\Delta$  of the unexecuted part (“*effective spread*”). It measures the *opportunity cost* that is associated with delayed execution on a per share basis.

For short time periods it is reasonable to model the two sides of the order book, the bid and the ask side, as being independent.<sup>14</sup> We therefore assume that the random variables  $V_T^\Delta$  and  $A_T^\Delta$  are conditionally independent given  $\Delta$ .<sup>15</sup> Then, we can write the *expected relative execution price* as

$$\begin{aligned} W(\Delta) &:= E[P^\Delta] = E[P^\Delta | \Delta] \\ &\stackrel{(2.1)}{=} E\left[\left(1 - \frac{V_T^\Delta}{N}\right) \cdot S_T^\Delta | \Delta\right] \\ &\stackrel{(*)}{=} E\left[\left(1 - \frac{V_T^\Delta}{N}\right) | \Delta\right] \cdot E[S_T^\Delta | \Delta] \\ &= \left(1 - \frac{E[V_T^\Delta]}{N}\right) \cdot E[S_T^\Delta] \\ &= \left(1 - \frac{E[V_T^\Delta]}{N}\right) \cdot \mu(\Delta), \end{aligned} \quad (2.2)$$

where  $\mu(\Delta) := E[S_T^\Delta]$  denotes the expected effective spread at the terminal time. To shorten notation, we might occasionally write  $\mu$  instead of  $\mu(\Delta)$ ; by construction,  $E[V_T^\Delta | \Delta] = E[V_T^\Delta]$  and  $E[S_T^\Delta | \Delta] = E[S_T^\Delta]$  since  $S_T^\Delta$  and  $V_T^\Delta$  are conditional on  $\Delta$ . The trader’s objective is now to find the *optimal display size*  $\Delta^*$ .

**Definition 1.** *The optimal display size  $\Delta^*$  is defined as*

$$\Delta^* = \arg \min_{0 \leq \Delta \leq N} \{W(\Delta)\}. \quad (2.3)$$

In order to guarantee that an optimal display size exists it is enough to assume some form of continuity of the dependence of the distribution of the total execution volume and ask price on the displayed part of the Iceberg order.

---

<sup>14</sup>Over short periods of time the two sides of the book are not tightly following each other as evidenced by the ubiquitous “widening” and “narrowing” of spreads. In the mid- and long run though, the two sides of the market will essentially co-move.

<sup>15</sup>Notice, by construction  $V_T^\Delta$  and  $A_T^\Delta$  refer to opposite sides of the book.  $V_T^\Delta$  denotes the execution volume on the traders (buy) side of the book, while  $A_T^\Delta$  denotes the best price on the opposite (ask) side.

## 2.1 Order Arrival Dynamics and Iceberg Execution Volume

The executed iceberg order volume is determined by the incoming order flow. Sell market orders execute against standing buy limit orders and improve the chance of execution while incoming buy limit orders add liquidity to the same side of the book and hence impede the chance of execution.

Modeling the full dynamics of *individual* order arrivals and cancelations would render the analysis of our model too complex. To enhance tractability, we use a *reduced-form model* of *aggregate* order flow. Specifically, order flows are aggregated into single submissions, effectively reducing our model to a 2-stage model: first (*aggregate*) limit orders arrive (or cancel); subsequently (*aggregate*) market orders arrive. Orders arrive according to a probabilistic dynamics that is independent of the stock price process; the aggregate market order volume arriving during the period  $[0, T]$  is denoted  $x \geq 0$  while the aggregate limit order volume at the submission and more competitive price levels is denoted  $y \geq 0$  and  $\hat{y} \geq 0$ , respectively.

Execution of standing limit orders by market orders is settled according to a set of priority or *precedence rules*. Almost all electronic trading venues give priority to orders submitted at more competitive prices and displayed orders have priority over hidden orders at the same price level. Orders with the same display status and submission price are usually served on a first-come-first-serve basis. We shall therefore apply the following ordered set of precedence rules:

- (i) Price Priority
- (ii) Display Priority
- (iii) Time Priority

The precedence rule identifies aggregate order flows, standing visible volume at the submission price level (denoted  $D$ ), the cancellation ratio ( $C$ ) of the standing orders at the top, and the hidden liquidity at the submission price level (denoted  $h$ ) as the determinants of the Iceberg execution volume.

At the time of submission the standing visible volume  $D$  has priority over the Iceberg Order; the displayed part  $\Delta$  has priority over the standing hidden volume ( $h$ ). On the other hand, by virtue of the time-priority rule, the hidden volume ( $h$ ) has priority over the Icebergs hidden part ( $N - \Delta$ ). Thus, after the limit orders have arrived the order volume with higher execution priority than the displayed Iceberg part is

$$Q^d := D(1 - C) + \hat{y}.$$

The total order volume with priority over the hidden part is

$$Q^h := Q^d + \Delta + h + y.$$



Thus, the sequence of execution priority reads  $(Q^d, h+y, N-\Delta)$  with the first entry representing the order volume of highest priority, and the execution volume  $V_T^\Delta$  is given by

$$V_T^\Delta = \begin{cases} 0 & x \leq Q^d \\ x - Q^d & Q^d < x \leq \Delta + Q^d \\ \Delta & \Delta + Q^d < x \leq Q^h \\ \Delta + x - Q^h & Q^h < x \leq Q^h + N - \Delta \\ \Delta + (N - \Delta) & Q^h + N - \Delta < x. \end{cases} \quad (2.4)$$

**Remark 2** (Order Aggregation and lower Execution Bound). *If we were to assume more general stochastic processes for the limit and market order arrivals using, for instance, poisson-arrival times  $n_x(t)$  and  $n_{\hat{y}}(t)$  and random order sizes  $(x_i)_{i \geq 0}$  and  $(\hat{y}_j)_{j \geq 0}$ , then a limit order's execution volume can be written*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left( \sum_{i=0}^{n_x(t)} x_i - \sum_{j=0}^{n_{\hat{y}}(t)} y_j \right) - D(1-C) \\ & \geq \left( \sum_{i=0}^{n_x(t)} x_i - \sum_{j=0}^{n_{\hat{y}}(t)} \hat{y}_j \right) - D(1-C) \\ & = x - \underbrace{(\hat{y} + D(1-C))}_{=Q^d} \\ & = x - Q^d \\ & = V_T^N \end{aligned} \quad (2.5)$$

where we have assume for simplicity that cancelations take place at end of trade-period. Hence, our model systematically under-estimates the “true” execution volume so order-aggregation yields a lower bound for order execution in limit order books. The conclusions for Iceberg Orders can be drawn in an analog way.

The execution volume  $V_T^\Delta$  is given in terms of the observable quantities  $D$ ,  $\Delta$ ,  $N$ , the cancellation ratio  $C$  and the random (unobservable) quantities  $h$ ,  $y$ ,  $\hat{y}$ , and  $x$ . Unobservable quantities are modeled as non-negative random variables.<sup>16</sup> In specifying their distributions we account for the important empirical observation that over shorter time periods order flows (market and limit) may be zero; see Table 3 in the appendix for selected estimates. The amount of standing hidden liquidity may also be zero.

In order to keep the model parameters to a minimum it is convenient to specify the distributions of random variables in terms of the probabilities of the variables taking the value zero and the conditional mean given they are positive. We therefore assume that the density functions of the flow variables  $x, y, \hat{y}$  and the hidden liquidity variable  $h$  take the form

<sup>16</sup>We denote random variables by lower case letters and deterministic quantities by capital letters.

$$f_y(s) = (1 - q) \cdot \mathbb{1}_{\{s=0\}} + \frac{q}{\beta} \cdot e^{-\frac{s}{\beta}} \cdot \mathbb{1}_{\{s>0\}} \quad (2.6)$$

$$f_{\hat{y}}(t) = (1 - \hat{q}) \cdot \mathbb{1}_{\{t=0\}} + \frac{\hat{q}}{\hat{\beta}} \cdot e^{-\frac{t}{\hat{\beta}}} \cdot \mathbb{1}_{\{t>0\}} \quad (2.7)$$

$$f_x(u) = (1 - p) \cdot \mathbb{1}_{\{u=0\}} + \frac{p}{\alpha} \cdot e^{-\frac{u}{\alpha}} \cdot \mathbb{1}_{\{u>0\}} \quad (2.8)$$

$$f_h(v) = (1 - r) \cdot \mathbb{1}_{\{v=0\}} + \frac{r}{\gamma} \cdot e^{-\frac{v}{\gamma}} \cdot \mathbb{1}_{\{v>0\}} , \quad (2.9)$$

where  $\mathbb{1}$  is the indicator function. The advantage is that with our choice of density functions the expected transaction volume can be given in closed form. This renders our model amenable at to some theoretical analysis. The proof of the following proposition is given in the appendix.

**Proposition 1** (Expected Execution Volume). *If  $p \cdot \alpha, N > 0$ , then*

$$E[V_T^\Delta] = \alpha p (1 - \hat{\beta}_r) e^{-\frac{D(1-c)}{\alpha}} \left\{ (1 - \beta_r)(1 - \gamma_r) \left( e^{-\frac{\Delta}{\alpha}} - e^{-\frac{N}{\alpha}} \right) + \left( 1 - e^{-\frac{\Delta}{\alpha}} \right) \right\} \quad (2.10)$$

where

$$\hat{\beta}_r := \hat{q} \frac{\hat{\beta}}{\alpha + \hat{\beta}} \quad \beta_r := q \frac{\beta}{\alpha + \beta} \quad \gamma_r := r \frac{\gamma}{\alpha + \gamma}. \quad (2.11)$$

*In particular, the expected execution volume is bounded by the expected amount of arriving market order volume:*

$$0 < E[V_T^\Delta] \leq p \cdot \alpha e^{-\frac{D}{\alpha}} \left( 1 - e^{-\frac{N}{\alpha}} \right) < p \cdot \alpha. \quad (2.12)$$

The first term in the curly brackets in (2.10) corresponds to the execution of the hidden part of the Iceberg Order. It depends on the parameters characterizing submission-level liquidity  $(\gamma, \beta)$  relative to the market order volume  $(\alpha)$ , the total order size  $N$ , and the display ratio relative to the expected market order volume. The terms  $(1 - \beta_r)$  and  $(1 - \gamma_r)$  reflect the loss in time-priority the hidden part suffers due to incoming visible and standing hidden orders at the submission price level, respectively. The quantity  $(1 - e^{-\frac{\Delta}{\alpha}})$  corresponds to the execution of the visible part; it only depends on  $\Delta$ .

The benchmark case  $\Delta = N$  captures the limit order case where only the second term matters. For Iceberg Orders both terms, the “displayed” and the “hidden” execution volume determine the overall (expected) execution volume. In what follows, we will always implicitly assume that  $\alpha \cdot p, N > 0$  holds. Otherwise, the Icebergs expected execution volume would be trivial, namely zero.

**Remark 3.** *As we can express the (expected) execution volume in terms of the ratios  $\hat{\beta}_r$ ,  $\beta_r$  and  $\gamma_r$ , the flows that interfere with the Iceberg’s execution enter the problem only relative to the market order flow  $\alpha$ .*

In order to understand how the exposure  $\Delta$  affects the trader’s execution performance, it is necessary to understand how the execution performance changes with changes of market parameters. The follows result is easily checked.

**Lemma 1.** *The following holds*

$$\frac{\partial W}{\partial \Delta} < 0 \quad \frac{\partial W}{\partial \alpha} < 0 \quad \frac{\partial W}{\partial \hat{\beta}_r} > 0 \quad \frac{\partial W}{\partial \beta_r} > 0 \quad \frac{\partial W}{\partial \mu} > 0. \quad (2.13)$$

## 2.2 Market Impact of Exposure and the Book-Imbalance

It has been empirically verified by many authors that the (visible) state of the order book carries information about future market dynamics.<sup>17</sup> In particular, Harris (1996, 2003) argues, that openly displayed limit orders pose a *free option* to other *rogue* or *parasitic* traders and that exposure may encourage “front-running” practices. In accordance with these empirical findings, we assume that, since they change the state of the *open* order book, openly displayed limit orders affect the market’s future behavior, a fact we refer to as *market impact of exposure* or simply *exposure impact*.

The way we model *exposure impact* is that the model parameters depend on the traders exposure choice (display size)  $\Delta$ . More precisely, our idea is to capture the impact of the display size  $\Delta$  through its impact on the top-of-the-book imbalance. The underlying assumption is that volume imbalance at the first price level is an indicator of future market movements, especially with respect to order flows and best ask prices.<sup>18</sup> We define the *relative* initial imbalance as

$$I_0 := \frac{D_{bid} - D_{ask}}{D_{bid} + D_{ask}},$$

where  $D_{bid}$  and  $D_{ask}$  denote the *initial* (visible) standing volume at the best bid and ask, respectively. Positive values represent bid-side liquidity excess, while negative represent sell-side excess. The imbalance as a function of the display size  $\Delta$  is then given by

$$I(\Delta) = \frac{D_{bid} - D_{ask} + \Delta}{D_{bid} + D_{ask} + \Delta} = \frac{I_0 - \Delta_r N_r}{1 + \Delta_r N_r} \quad (2.14)$$

where  $\Delta_r := \frac{\Delta}{N}$  denotes the *display ratio* and  $N_r := \frac{N}{D_{ask} + D_{bid}}$  is the *volume ratio*, i.e, the percentage of volume a fully displayed Iceberg order would add to the book. Thus, the impact of the display size on the imbalance is felt trough the proportion of visible orders added to the top of the book.

*Market Impact* of order exposure is now realized by allowing the free model parameters, the (conditional) probabilities  $p, \hat{q}, q, r$  and conditional means  $\alpha, \beta, \hat{\beta}, \gamma$  as well as  $\mu$ , to depend on the imbalance  $I(\Delta)$ . To denote this dependency we may henceforth write  $\alpha(I)$  or  $\alpha_I$  and in the same fashion for the other parameters. The trader’s expected execution price (2.2) can thus be

---

<sup>17</sup>For instance, see Biais et al. (1995); Huang and Stoll (1994); Cao et al. (2009); Chordia et al. (2002); Beber and Caglio (2005); Ellul et al. (2003); Harris and Panchapagesan (2005); Ranaldo (2004); Griffiths et al. (2000).

<sup>18</sup>Findings in Hellström and Simonsen (2006); Pascual Gasco and Veredas (2008); Cao et al. (2009); Chordia et al. (2002); Huang and Stoll (1994) suggest that the order book imbalance  $I$  captures most of the explanatory power of the order book state. Cao et al. (2009) Cao et al. (2009), Simonsen et al. (2003) Hellström and Simonsen (2006) and Pascual et al. (2004) Pascual Gasco and Veredas (2008) report that most of the imbalance’s explanatory power concentrates on the best quotes.

expressed as

$$W(\Delta, I(\Delta)) := W\left(\Delta, \alpha(I(\Delta)), \beta_r(I(\Delta)), \hat{\beta}_r(I(\Delta)), \mu(I(\Delta))\right). \quad (2.15)$$

We notice that the expected execution price depends directly on the display size through a loss in time priority (*priority impact*) and indirectly through its impact on imbalances and hence flow parameters (*exposure impact*).

### 2.3 Analytical Discussion

Taking the total derivative of  $W$ , changes in the expected execution price due to infinitesimal changes in the display size can be decomposed into an *exposure-impact* and a *priority-impact* term:

$$\begin{aligned} \frac{d}{d\Delta} W &= I'(\Delta) \left( \frac{\partial \alpha}{\partial I} \frac{d}{d\alpha} + \frac{\partial \beta_r}{\partial I} \frac{\partial}{\partial \beta_r} + \frac{\partial \hat{\beta}_r}{\partial I} \frac{\partial}{\partial \hat{\beta}_r} + \frac{\partial \mu}{\partial I} \frac{\partial}{\partial \mu} \right) W - \left( -\frac{\partial W}{\partial \Delta} \right) \\ &:= M_{Market} - M_{Priority}. \end{aligned} \quad (2.16)$$

Since the cumulative market order flow volume ( $\alpha$ ) enters the definition of  $W$  explicitly and implicitly through its impact on  $\hat{\beta}_r$  and  $\beta_r$ , we take the total instead of partial derivative with respect to  $\alpha$ . Of course, the optimal display size  $\Delta^*$  satisfies

$$\Delta^* = \begin{cases} N & \text{if } M_{Market}(\Delta) < M_{Priority}(\Delta) \\ 0 & \text{if } M_{Market}(\Delta) > M_{Priority}(\Delta) \end{cases} \quad \text{for all } \Delta \in [0, N]. \quad (2.17)$$

Moreover, if the mapping  $\Delta \rightarrow W(\Delta)$  is strictly convex and  $\Delta^* \in (0, N)$  then the optimal display size is characterized by the fact that the priority-impact exactly outweighs the exposure-impact:

$$M_{Market}(\Delta^*) = M_{Priority}(\Delta^*).$$

Closed-form solutions for the optimal display size will not be available in general, due to the highly non-linear dependence of the model parameters on the display size. We shall therefore confine our theoretical analysis to asymptotic and monotonicity considerations and identify situations where one of the conditions in (2.17) is satisfied.

#### 2.3.1 Absence of market impact

As the priority-rules favor visible over hidden liquidity one expects the priority-term ( $M_{Priority}$ ) to systematically reduce transaction costs, so its contribution to the total differential should be negative; Lemma 1 confirms this assertion. Matters are less clear cut for the market impact term ( $M_{Market}$ ). Depending on how the market reacts to changes in the imbalance/exposure, the term may either penalize or reward exposure. We first identify a situation where markets reward exposure full display is optimal.

**Corollary 1.** *Assume that, on average, liquidity demand increases and liquidity supply decreases with larger book-imbalance and assume that the price (on average) moves in the direction*

of excess-imbalance, i.e.  $-\frac{\partial\alpha}{\partial I}, \frac{\partial\beta_r}{\partial I}, \frac{\partial\hat{\beta}_r}{\partial I}, \frac{\partial\mu}{\partial I} \leq 0$ . Then

$$\Delta^* = N \tag{2.18}$$

*Proof.* In view of (2.14) and Lemma 1, we have  $I'(\Delta) > 0$  and  $-\frac{dW}{d\alpha}, \frac{\partial W}{\partial\beta_r}, \frac{\partial W}{\partial\hat{\beta}_r}, \frac{\partial W}{\partial\mu} > 0$ . Now one checks that in the case  $-\frac{\partial\alpha}{\partial I}, \frac{\partial\beta_r}{\partial I}, \frac{\partial\hat{\beta}_r}{\partial I}, \frac{\partial\mu}{\partial I} \leq 0$ , the market-impact-term ( $M_{Market}$ ) in (2.16) is negative. Since the priority term ( $M_{Priority}$ ) is negative, the total derivative in (2.16) is negative and hence  $\Delta^* = N$ .  $\square$

In the preceding case, exposing the full order allows the trader to benefit from the gain in time-priority to the full extent.<sup>19</sup> As we shall see below, though, in most cases the market impact term ( $M_{Market}$ ) - instead of rewarding - penalizes order exposure.<sup>20</sup> This suggests that time-priority-gain and exposure-impact are counter-acting mechanisms. Hence, at this point we anticipate that there will be some trade-off between these two impact-contributions that determines the optimal display size.

### 2.3.2 Asymptotics

We are now going to identify (limiting) situations where where full, respectively no display is optimal. As it turns out, the results strongly depend on how prices react to changes in imbalances. We shall therefore distinguish the following two cases:

- (I)  $\frac{\partial\mu}{\partial I} > 0$
- (II)  $\frac{\partial\mu}{\partial I} < 0$ .

Since the imbalance  $I(\Delta)$  is an increasing function of the display size ( $\Delta$ )

$$\text{sign}\left(\frac{\partial\mu}{\partial I}\right) = \text{sign}\left(\frac{\partial\mu}{\partial\Delta}\right). \tag{2.19}$$

The following proposition states that in markets where prices “move away” from *exposure*, large traders should hide their trading intentions as the *opportunity costs* associated with paying high prices at terminal time can be significant. On the other hand, exposure-rewarding markets eliminate the *opportunity risk* associated with partial execution. Thus traders can expose their trading intentions.

**Proposition 2.** *Suppose that all functions depend sufficiently smoothly only order imbalances. Then there is a (sufficiently large) order size  $N_d$  such that for any  $N > N_d$  the respective*

---

<sup>19</sup>We call this regime “Absence of Market Impact”, since this form of market impact is not adversely affecting the trader’s execution performance.

<sup>20</sup>Consistent with this, earlier research has found that parasitic traders are front-running large visible limit orders and that prices may be adversely affected by the exposure decision; markets will penalize order exposure at least to some extent; see, e.g. Harris (1996, 2003); Bessembinder et al. (2009).

optimal display size  $\Delta^*(N)$  obeys

$$\Delta^*(N) = \begin{cases} 0 & \frac{\partial}{\partial I}\mu > 0 \\ N & \frac{\partial}{\partial I}\mu < 0 \end{cases} . \quad (2.20)$$

*Proof.* We can multiply (2.16) with  $N$  and rewrite it as

$$\begin{aligned} N \frac{d}{d\Delta} W &= N \cdot I'(\Delta) \left( \frac{\partial \alpha}{\partial I} \frac{d}{d\alpha} + \frac{\partial \beta_r}{\partial I} \frac{\partial}{\partial \beta_r} + \frac{\partial \hat{\beta}_r}{\partial I} \frac{\partial}{\partial \hat{\beta}_r} + \frac{\partial \mu}{\partial I} \frac{\partial}{\partial \mu} \right) W(\Delta) - N \cdot \left( -\frac{\partial W}{\partial \Delta} \right) \\ &= N \cdot \left\{ \frac{\partial W}{\partial \Delta} + I' \cdot \left( \frac{\partial \alpha}{\partial I} \frac{d}{d\alpha} + \frac{\partial \beta_r}{\partial I} \frac{\partial}{\partial \beta_r} + \frac{\partial \hat{\beta}_r}{\partial I} \frac{\partial}{\partial \hat{\beta}_r} \right) W \right\} (\Delta) + N \cdot I' \cdot \frac{\partial \mu}{\partial I} \cdot \frac{\partial}{\partial \mu} W(\Delta) \end{aligned}$$

Next, observe that  $N \cdot W = (N - E[V_T^\Delta]) \cdot \mu(\Delta)$ . Hence the first term of the equation above equals

$$I'(\Delta) \left( \frac{\partial \alpha}{\partial I} \frac{d}{d\alpha} + \frac{\partial \beta_r}{\partial I} \frac{\partial}{\partial \beta_r} + \frac{\partial \hat{\beta}_r}{\partial I} \frac{\partial}{\partial \hat{\beta}_r} + \frac{\partial \mu}{\partial I} \frac{\partial}{\partial \mu} \right) (N - E[V_T^\Delta]) \mu(\Delta)$$

Now, if  $I$  is a smooth enough function of the display size and  $\beta_r, \hat{\beta}_r$  and  $\alpha$  are smooth functions of order imbalances, this term is bounded. Hence, since  $I' > 0$  and  $\frac{\partial}{\partial \mu} W > 0$  we see that for large positions the sign of  $\frac{d}{d\Delta} W$  is determined by the sign of  $\frac{\partial \mu}{\partial I}$  and hence the result follows.  $\square$

According to (2.16), the market impact-term ( $M_{Market}$ ) depends strongly on the sensitivity of the imbalance with respect to the display sizes. This suggests, that adding/exposing only small orders to already large volumes at the top-of-the-book, or already large imbalances, will not alter the imbalance substantially. In this case, the trader suffers only minor exposure-impact related impact, while she fully benefits from the gain in time-priority. Therefore, according to the next proposition, the trader is better-off exposing her trading intentions.

**Proposition 3** (Imbalance, Standing Liquidity and Optimal Display Size). *(i) For large or small enough (initial) depth  $D_{ask}$  at the opposite side of the book, the optimal display size obeys*

$$\Delta^* = N. \quad (2.21)$$

*(ii) Assume fixed initial depth  $D_{bid}$  at the same side. For sufficiently negative or positive imbalances, the optimal display strategy obeys*

$$\Delta^* = N. \quad (2.22)$$

*(iii) For sufficiently large (initial) depth  $D_{bid}$  on the same side of the book, the optimal display size obeys*

$$\Delta^* = \begin{cases} 0 & \frac{\partial \mu}{\partial I} > 0 \\ N & \frac{\partial \mu}{\partial I} < 0 \end{cases} . \quad (2.23)$$

*Proof.* In view of (2.14), we have  $I'(\Delta) = \frac{D_{ask}}{(D_{bid} + D_{ask} + \Delta)^2}$ . Hence it follows from (2.16) that the market impact term vanishes and the priority term dominates for  $D_{ask} \rightarrow \infty$  and  $D_{ask} \rightarrow 0$ . This proves the first assertion. The second assertion follows because for a given bid side depth, large imbalances occur if  $D_{ask} \rightarrow \pm\infty$ . As for the third assertion, let us assume that  $\mu I > 0$ ;

the other case follows by analogy. As in the proof of the previous proposition, we write the expression in (2.16) as

$$\begin{aligned}
\frac{d}{d\Delta}W &= M + I'(\Delta)\frac{\partial\mu}{\partial I}\frac{\partial}{\partial\mu}W \\
&> M + I'(\Delta)(N - \alpha(\Delta))\frac{\partial\mu}{\partial I} \\
&\stackrel{(*)}{=} me^{-\frac{D_{bid}(1-C)}{\alpha}} + \frac{D_{ask}}{(D_{bid} + D_{ask} + \Delta)^2}(N - \alpha(\Delta))\frac{\partial\mu}{\partial I},
\end{aligned} \tag{2.24}$$

where we abbreviated the left term by  $M$ . In (\*) we used the fact that all terms of  $M$  have a common factor, namely  $e^{-\frac{D_{bid}(1-C)}{\alpha}}$ . This can be easily seen by simply checking the respective derivatives of  $W$  (2.2). Due to exponential decay for  $D_{bid} \rightarrow \infty$  the second term will dominate for sufficiently large  $D_{bid}$ . Since  $\frac{\partial\mu}{\partial I} > 0$  one sees that  $\frac{\partial W}{\partial\Delta} > 0$  for all  $0 \leq \Delta \leq N$  and hence the assertion follows.  $\square$

**Remark 4.** *The proof of (iii) hinges on the assumption, that the distribution of aggregate market order arrival volumes has exponential decay, see (2.9). We have chosen the exponential form out of convenience and analytical tractability. In general, one might expect the distribution to be fat-tailed. The proof suggests, when the distribution decays weaker than  $\frac{1}{1+x^2}$ , the conditions for (iii) might be violated and in the limit  $D_{bid} \rightarrow \infty$  the time-priority-term might prevail. In this case, we might obtain  $\Delta^* = N$ , irrespective of the monotonicity of  $\mu$  with respect to  $I$ .*

The previous proposition studied the asymptotic effect of the (initial) standing liquidity on the trader's optimal display choice. For fixed initial best bid depth ( $D_{bid}$ ), we found that if the best ask depth ( $D_{ask}$ ) is sufficiently high (high imbalance-skew), all contributions but the time-priority contributions ( $M_{Priority}$ ) vanish. This is because adding additional liquidity to an already highly skewed book contributes only marginally to the imbalance. The situation is different for the best bid ( $D_{bid}$ ). For large  $D_{bid}$ , instead of the priority-term, the price-impact term dominates and the optimal exposure strategy depends on the direction of market impact. The reason that the results for best bid and best ask depths is that our trader is a buyer. Hence, the standing volume  $D_{bid}$  is important (it is the queue that needs to be filled before the Iceberg order can get served) while  $D_{ask}$  only affects the order imbalance.

We close this section with a brief comment on the dependence of display sizes on flow dynamics.

**Proposition 4** (Liquidity Flow and Optimal Display Size). *For sufficiently strong liquidity competition, i.e.  $\inf_I \hat{\beta}_r$  sufficiently large and  $\frac{\partial\hat{\beta}_r}{\partial I} \rightarrow 0$  as well as for sufficiently weak liquidity demand, i.e.  $\sup_I \alpha$  sufficiently small, the optimal display size obeys*

$$\Delta^* = \begin{cases} 0 & \frac{\partial\mu}{\partial I} > 0 \\ N & \frac{\partial\mu}{\partial I} < 0 \end{cases}. \tag{2.25}$$

*Proof.* For  $\inf_I \hat{\beta}_r \rightarrow 1$  and  $\frac{\partial\hat{\beta}_r}{\partial I} \rightarrow 0$  it is easily checked that

$$\frac{\partial W}{\partial\Delta} = 0 \qquad \frac{dW}{d\alpha} = 0 \qquad \frac{\partial W}{\partial\beta_r} = 0 \qquad \frac{\partial W}{\partial\hat{\beta}_r} = 0. \tag{2.26}$$

Hence, applying the same reasoning as above it follows from (2.16) that the sign of  $\frac{\partial\mu}{\partial I}$  deter-

mines whether full or no display is optimal. As for the second assertion notice that  $\sup_I \alpha \rightarrow 0$  implies

$$\frac{\partial W}{\partial \Delta} = 0 \quad \frac{dW}{d\alpha} = 0 \quad \frac{\partial W}{\partial \beta_r} = 0 \quad \frac{\partial W}{\partial \hat{\beta}_r} = 0 \quad (2.27)$$

so the same arguments as before apply.  $\square$

### 2.3.3 Execution volume

Besides *execution costs*, *execution volume* is a common benchmark for execution performance. In this section we compare optimal expose strategies for volume and transaction-costs-traders. A *volume-trader* maximizes the expected execution volume  $E[V_T^\Delta]$  while a *transaction-cost trader* minimizes the expected execution costs. Their respective optimal display sizes are denoted by  $\Delta_V^*$  and  $\Delta_W^*$ .

**Proposition 5.** *Let  $N > 0$ . Then*

$$\Delta_V^* \begin{cases} \geq \Delta_W^* & \frac{\partial \mu}{\partial I} > 0 \\ \leq \Delta_W^* & \frac{\partial \mu}{\partial I} < 0 \end{cases} \quad (2.28)$$

*Proof.* Let  $\frac{\partial \mu}{\partial I} > 0$ ; the opposite case follows analogously. Let us assume to the contrary that there exists maximizers  $\Delta_W^* > \Delta_V^*$ . Then, by definition of  $W$  given in (2.2) and setting  $\mathbf{V}(\Delta) := E[V_T^\Delta]$ , one has

$$1 > \frac{W(\Delta_W^*)}{W(\Delta_V^*)} = \frac{(N - \mathbf{V}(\Delta_W^*))\mu(\Delta_W^*)}{(N - \mathbf{V}(\Delta_V^*))\mu(\Delta_V^*)}$$

Since  $\mu(\Delta) = \mu(I(\Delta))$  is strictly increasing in  $\Delta$ , this is equivalent to

$$\frac{N - \mathbf{V}(\Delta_V^*)}{N - \mathbf{V}(\Delta_W^*)} > \frac{\mu(\Delta_W^*)}{\mu(\Delta_V^*)} > 1$$

from which one gets that  $\mathbf{V}(\Delta_V^*) < \mathbf{V}(\Delta_W^*)$ . This contradicts the fact that  $\Delta_V^*$  maximizes expected execution volume.  $\square$

The reason a volume-trader displays more is that *opportunity* costs associated with unfilled orders are not important to her. She solely looks at maximizing the total execution volume at the submission price level, in-effect ignoring that she might have to pay an additional premium in order to finish the trade.

### 2.3.4 Concluding remarks

Our analytical results are in line with the empirical findings in Bessembinder, Panayides, and Venkatamaran (2009), especially their cross-sectional regression analysis<sup>21</sup>. The findings of

---

<sup>21</sup>Their empirical results on trader's exposure strategies, reveals that the decision to expose as well as the exposure-magnitude strongly depend on the prevailing order book state, order arrival activity and order attributes. In particular, they report significant t-statistics for the order size (27.64), same-side (-3.70) and



Propositions 2, 3 and 4) indicate that - *ceteris paribus* - traders hide more when the order size is large, the best bid depth is large, the market order volume is low and the limit order flow competition is strong for “*away-drifting markets*”. Proposition 5 is in line with another main result of Bessembinder et al. (2009), namely the fact that exposure shortens time-to-completion, amounting to higher execution volume for fixed trading horizon, while hiding generally reduces transaction costs. This is exactly what Proposition 5 predicts if  $\frac{\partial \mu}{\partial I} > 0$ . The fact that our model shows strong consistency with the empirical findings of Bessembinder et al. (2009) for “*away-drifting markets*” suggests that realistically  $\frac{\partial \mu}{\partial I} > 0$ . In the sequel, we provide strong empirical evidence that this is indeed the case.

### 3 Model Calibration

Our model hinges upon the assumption, that order flows and prices are affected by the trader’s exposure decision through its impact on order imbalances. In order to calibrate the model we thus need to estimate the functions

$$I \rightarrow q_I, \hat{q}_I, p_I, \beta_I, \hat{\beta}_I, \alpha_I, r_I, \gamma_I, \mu_I. \quad (3.1)$$

This amounts to estimating the impact of openly displayed limit orders. In the theoretical and empirical literature so far, market impact considerations have been mainly undertaken with respect to *market orders*; cf. the literature on optimal liquidation (Almgren and Chriss (1999); Obizhaeva and Wang (2005); Alfonsi et al. (2010); Alfonsi and Schied (2010) and others). Market impact of *passive* (limit, Iceberg) orders is different from the market impact of *aggressive* (market) orders. Aggressive orders are executed against standing liquidity and hence incur an instantaneous price impact. By contrast, passive orders incur execution-risk, the latter being essentially governed by order flow. This suggests that impact of a (displayed) limit order should be captured through its impact on future orders. Consistent with Harris (1996, 1997), our estimates of the model parameters (3.1) indicate that market impact of exposure is largely due to the presence of *parasitic traders*. We find only weak evidence that exposure impact is caused by *impatient* trader’s retreating from the market as reported in Moinas (2010).

Our estimates of the flow parameters are used to estimate transaction costs and optimal display strategies. In the sequel we provide optimal display strategies for selected stocks from the US S&P-500-index and benchmark the performance of Iceberg Orders against that of plain limit orders. Our results suggest that Iceberg Orders may help to significantly lower transaction costs.

---

opposite-side (-3.40) depth, same-side hidden depth (3.84), order book imbalance (-3.72), *waiting time* (2.03) and trade size (-5.16). These results indicate - *ceteris paribus* - that traders hide more when the order size is large, the quoted liquidity depth is small, the book-imbalance is skewed towards the opposite side of the market and the market order volume is high and the liquidity competition is strong. Similar empirical results have been reported by De Winne and D’Hondt (2007)

### 3.1 The Data Set

Our estimates are based on Message Level data from the Instinet (INET)<sup>22</sup> market for the period of January and February 2009. This data-set provides messages for every order entry, including modification, cancelation, submission and execution. The messages contain order identification number, time stamps, modification/submission/cancelation/execution size, as well as a flag marking the side of the book (buy or sell). This way, we were able to track every order until cancellation/execution and to re-construct the visible order book.

In order to estimate the dependence of the model parameters on imbalances we used a sample of *non-intersecting*  $\Delta t$ -periods during 9:30 and 15:30 hrs for which - for each realization of the initial imbalance  $I$ - we record the cumulative flow volumes  $(x_I, \hat{y}_I, y_I)$ , standing hidden volume  $(h_I)$  as well as the effective spread at the respective terminal time, and constructed the Maximum-Likelihood-estimates (MLE) for the corresponding flow  $(q_I, \hat{q}_I, p_I, \beta_I, \hat{\beta}_I, \alpha_I)$  and price  $(\mu_I)$  parameters.<sup>23</sup>

The INET-data set does not send messages for modification and cancellation of *hidden orders* which renders the reconstruction of the hidden volume  $h_I$  at a given price level incomplete. In order to obtain proper estimates for the hidden parameters  $(r_I, \gamma_I)$ , we use NASDAQ-Model-View-data instead.<sup>24</sup> At each price level, this data set provides full minute-by-minute snapshots of the market's *consolidated* visible and hidden depth for NASDAQ, NYSE and AMEX-listed stocks. This includes the following selection of liquid, high-tech S&P500-stocks from the INET-exchange: <sup>25</sup> Cisco Systems Inc. (CSCO), Dell Inc. (DELL), eBay Inc. (EBAY), Hewlett-Packard Company (HPQ), Microsoft Corp. (MSFT) and Oracle Corp. (ORCL).

### 3.2 Parameter Estimation and Market Impact of Limit Orders

For the estimation we assume a discretization of the imbalance interval of 0.15 points in a range between -0.7 and 0.7.<sup>26</sup> For each realization of  $I$ , we construct the respective Maximum Likelihood Estimates (MLE) for the model parameters  $q_I, \hat{q}_I, p_I, \beta_I, \hat{\beta}_I, \alpha_I, r_I, \gamma_I, \mu_I$ , see (B.5). In order to obtain smooth functional representations, additionally we apply a *cubic weighted Ordinary Least Squares* (wOLS)<sup>27</sup> on the point estimates. We use *weighted* OLS in order to properly account for possible *heteroscedacity*-effects.<sup>28</sup> For  $\Delta t = 30s$ , examples of estimated

---

<sup>22</sup>INET was one of the largest and first recognized electronic communications network (ECN) in the US, starting its first electronic trading venue in 1969. It was bought by NASDAQ in 2005. As of the last quarter of 2008 INET holds 5% share of the total US market in traded equity volume.

<sup>23</sup>We note the reader, that we use the same notation for the "true" parameter values and the empirical MLE-estimates, as we supposed to be clear from the context what we mean.

<sup>24</sup>We acknowledge that we simply *assume* that the NASDAQ data is a good proxy for hidden liquidity at INET.

<sup>25</sup>We only consider trading activity as recorded on the INET stock exchange. Note however, that these stocks are traded simultaneously on multiple venues as the US equities market is fragmented and there is no centralized trading platform.

<sup>26</sup>Samples beyond this imbalance range are sparse and thus don't gather sufficient statistics.

<sup>27</sup>Cubic polynomials are the first choice for model/regression functions that are nonlinear and at the same time account for possible asymmetric relationships.

<sup>28</sup>Note that extremely large positive or negative imbalances  $I$  may occur less frequently than smaller ones, since in general they afford more liquidity. Thus the sample sizes and thereby the sample variances may vary

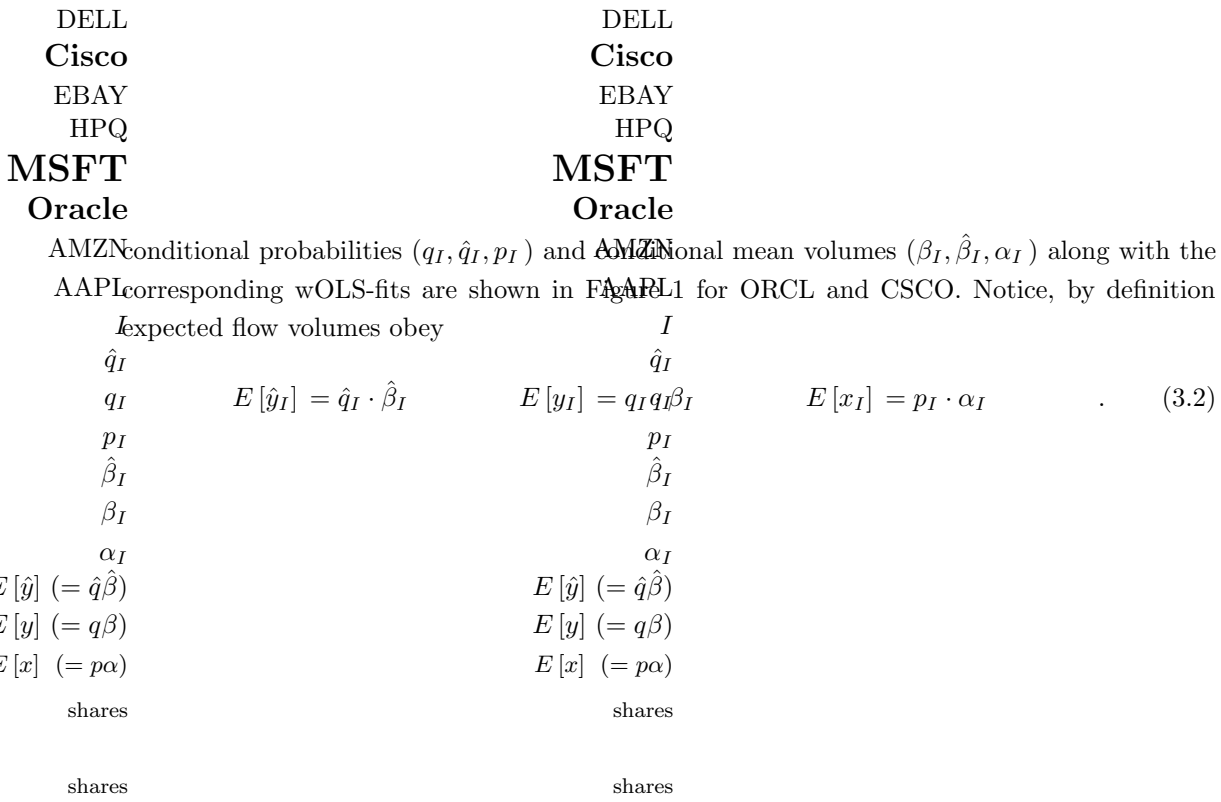


Figure 1: Example of Flow Parameter Estimation for Oracle and Cisco as a function of imbalance for  $\Delta t = 30s$ . Observe that the “front-running” limit order flow  $\hat{y}$  shows strong dependence on the order books imbalance. The dependence of market order flow and limit order flow at the submission price level is comparably marginal.

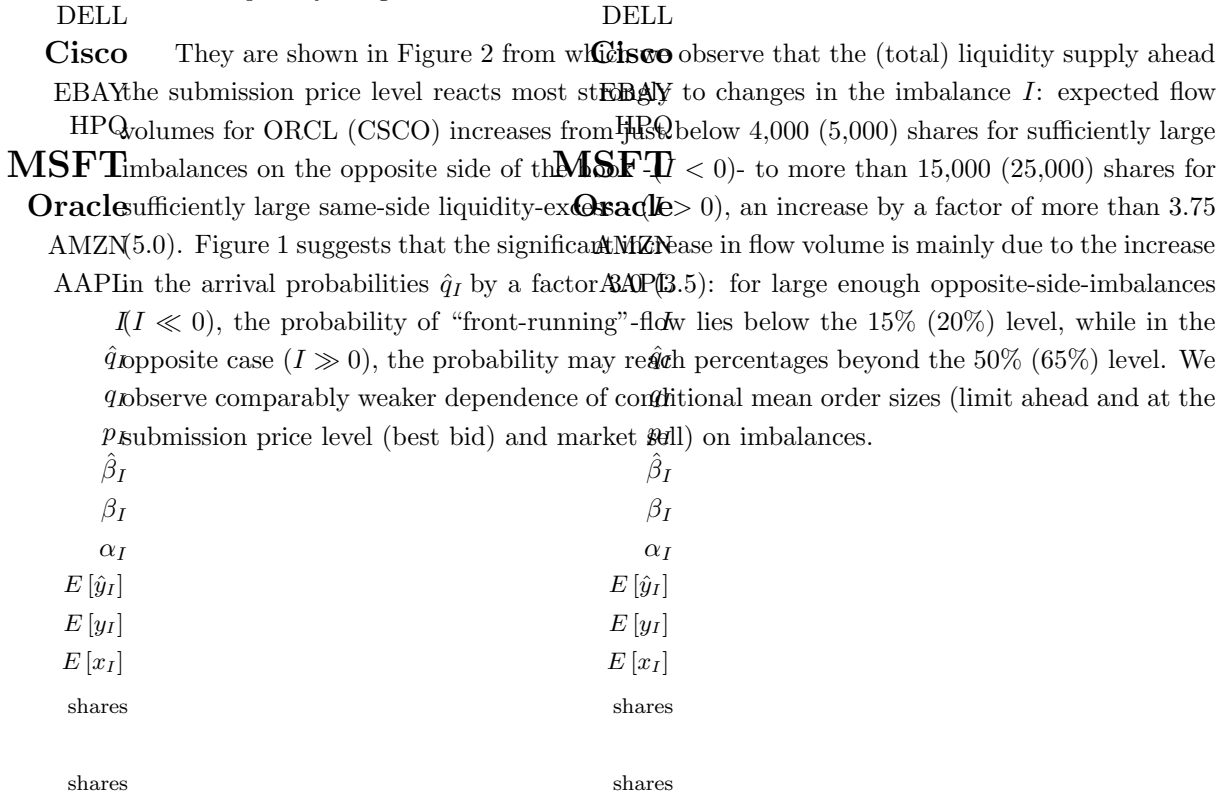


Figure 2: Expected aggregate conditional order flow volume.

Besides the order flow dynamics at the same side of the market, the best ask price  $A_T$  at the terminal time is a key determinant of the trader’s execution cost. In the same fashion as for

---

conditional on  $I$ .

<u>PSfrag replacements</u>	<u>PSfrag replacements</u>
CSCO	CSCO
ORCL	ORCL
$I$	$I$
$E[A_T   I]$	$E[A_T   I]$
$\Delta t = 3s$	$\Delta t = 3s$
$\Delta t = 10s$	$\Delta t = 10s$
$\Delta t = 30s$	$\Delta t = 30s$

Figure 3: Price Impact of the Order Book Imbalance  $I$  for Oracle and Cisco for  $\Delta t = 30s$ . The expected conditional price best ask price  $E[A_T | I]$  (in bps) against the (initial) relative imbalance  $I$ . Note, that for the buy-trader the “effective spread“ and the best ask price obey  $S_T = \frac{A_T^\Delta - B_0}{B_0}$ .

the flow properties, we thus estimated its (conditional) mean values  $E[A_T | I]$ . In Figure 3, we provide the estimate results for CSCO and ORCL for the case of  $\Delta t = 30s$ . We observe, that the stronger the (initial) order book imbalance the stronger the price change in the direction of the imbalance-excess. The expected 30-seconds future price change for CSCO (ORCL) equals about 4 (2) basis points for relative imbalances beyond 0.5. As the average spread for this stock lies at about 6 (5)<sup>29</sup>, we can conclude that the price increment is significant. We report *asymmetry* in that the imbalance-impact on the price is not totally symmetric with respect to the origin: For zero imbalance, one still expects a positive price drift, nevertheless still within the range of the stock’s average spread. Moreover, in terms of basis points price change for positive imbalances - bid-side excess - is more pronounced than for negative imbalances. While as for Cisco, we observed an expected price change of about 3 (1.7) basis point for  $I = 0.5$ , the price change for the negative imbalance of  $I = -0.5$  is about -1 basis points (for both). Figures 6 and 7 in the appendix provide additional flow and price estimates for the stocks of Apple, Amazon, Ebay, Hewlett-Packard, Dell and Microsoft and for the periods  $\Delta t = 3s, 10s, 30s$ .

### 3.3 Correlation Analysis

Table 2 shows estimated correlations between the (initial) imbalance and the conditional expected order flows and price increments. We observe, that apart from some market order flow, the null hypothesizes that the imbalance and the respective conditional expectations of flow volumes and the price increments are independent can be rejected at the 0.001 level of significance in most cases. The correlation estimates indicate, that the best ask price and the front-running flow ( $\hat{y}$ ) react most strongly to changes in the imbalance; we report throughout significant correlation estimates.<sup>30</sup> For stocks like Apple (AAPL) and Amazon (AMZN) that on average trade at larger spreads our estimates indicate less imbalance-related correlation with market’s future flow and price. The substantial positive correlation for  $E[\hat{y}_I]$  suggests that imbalance

<sup>29</sup>See table 1 for details.

<sup>30</sup>The correlation estimates are throughout double-digit in the case of 1-tick spread stocks

on the trader’s side triggers “front-running” (bid) limit order flow. Indeed, we see from Figure 6 that for  $\Delta t = 3s$  and for sufficiently large imbalances on the opposite (same) side of the book, the expected liquidity supply flow ahead the submission price level can increase to more than 400% (drop down to below than 40%) as compared to the volume in the balanced case, i.e. for  $I \approx 0$ . This finding confirms Harris’ *parasitic trader*-scenario of exposure-impact: Exposure causes these traders to *undercut* the limit order and post at more aggressive prices and is consistent with the empirical literature on *order aggressiveness*.<sup>31</sup> On the other hand, we only find weak indication for the alternative *scenario*: Exposure (imbalance-skew) causes *passive* traders to retreat from the market, as we don’t observe significant variations in the market order flow with changes in the imbalance; however, this effect might somewhat be still significant in the case of large opposite-side imbalance-excess, as for some stocks Figure 6 reports a drop in flow volume from opposite-side excess (negative imbalance) to same-side-excess (positive imbalance). Finally, the correlation estimates indicate that generally prices move against the trader the larger the imbalance on the trader’s side, increasing the trader’s “opportunity” costs.<sup>32</sup>

Let us conclude this section with a brief discussion of the temporal structure of the correlation analysis, starting first for stocks that trade 1-tick spreads (Table 2). Except for Dell (DELL) and Ebay (EBAY), we observe a decay in market impact over time. The “font-running” flow  $\hat{y}$  and price  $A_T$  correlations tend to decrease with time. For example, for Cisco (CSCO) the price correlation drops from 0.23 for a 3s-period to 0.12 for a 30s-period. For Hewlett-Packard (HPQ) it drops from 0.19 to 0.06. Generally, flow show less decay in time than price correlations. While both Dell and Ebay show correlation peaks for the “front-running” order flow at 10s-periods, only Dell shows a correlation peak for the price process for the same period. However price-correlations for Ebay between the 3s- and the 10s- period differ only marginally. A possible explanation for the correlation peak at mid-time periods for Ebay and Dell is that they trade slower and thus need longer to generate a reaction to changes in the order book (i.e. imbalance) than more actively traded stocks; see also Table 3 where we reported probabilities of order arrivals for different time periods, indicating how actively the stocks are traded. Among all stocks considered, Ebay and Dell exhibit the smallest event-probability over all time periods, indicating that these stocks are less actively traded than the others. We obtain a similar behavior for the stocks with wider spreads. While for Apple the correlations decay for both small (one tick) (a) and large (two ticks) spreads (b), Amazon shows a significant correlation peak at the  $\Delta t = 10s$  period for small spreads. Again, applying the same reasoning, Table 3 suggests that Amazon is less actively traded than Apple. These findings are consistent with the empirical literature surrounding the issue of *information content of the limit order book*, where the order-book-state’s impact on the evolution of future market behaviour is studied, see Hellström and Simonsen (2006); Pascual Gasco and Veredas (2008); Cao et al. (2009); Chordia et al. (2002); Huang and Stoll (1994). Moreover, since the imbalance is a function of the openly displayed part of (limit) orders, our empirical findings as well add to the research surrounding

---

<sup>31</sup>We refer to Ranaldo (2004); Bessembinder et al. (2009); Griffiths et al. (2000); Biais et al. (1995); Chordia et al. (2002); Beber and Caglio (2005); note that “front-running” flow is equivalent to “aggressive” order placement.

<sup>32</sup>Note that the trader has to complete the unexecuted Iceberg order against the best (relative) ask price  $A_T$  according to the model setup. Thus the more the price moves away, the larger the transaction costs  $W$ . Costs that are related to partial executions are often referred to as “*opportunity costs*”.

the *market impact of limit orders*.<sup>33</sup> Namely, order exposure trigger’s *price-improving* and competing limit order flow on the same side of the book and (on average) causes the best quoted prices to move away from the trader’s position.

### 3.4 Optimal Exposure Strategies: Calibration Results

Our estimation results allow us to compute the optimal exposure decision ( $\Delta^*$ ) and to study the dependence of  $\Delta^*$  on the model parameters within our theoretical model. In order to compare the display decisions for different order sizes and stocks, we introduce the optimal display ratio  $\Delta_r^* := \frac{\Delta^*}{N}$ . Figures 8 and 9 show the results for the case  $\Delta t = 10s$ . The optimal display ratios are drawn with respect to the initial buy-side depth ( $D_{bid}$ ) and imbalance ( $I_0$ ). Optimal exposure strategies close to 1 (i.e. full exposure) are colored red; exposure strategies close to 0 (i.e. zero exposure), they are colored blue.<sup>34</sup> According to our data sampling, initial imbalances ( $I_0$ ) range between  $-0.7$  and  $0.7$ . The initial same-side depth ( $D_{bid}$ ) ranges have been chosen so as to include typical average best bid depths for the respective stocks; see Table 1 in the Appendix. For order sizes ( $N$ ), we have chosen three values for each stock: a small, an intermediate and a large one.

Throughout the range of stocks, we observe that for small order sizes (left-hand-side of Figure 8 and 9), the optimal exposure strategy is generally *full display* (i.e.  $\Delta_r^* = 1$  for most values of  $D_{bid}$  and  $I_0$ ). Exceptions for some stocks occur for large initial same-side depth ( $D_{bid}$ ), as in the case of Cisco, Dell, Ebay and Oracle. However, as our markets obey  $\frac{\partial \mu}{\partial I} > 0$ <sup>35</sup>, this is consistent with Proposition 3. For intermediate order sizes the optimal display ratio  $\Delta_r^*$  has decreased significantly across the parameter range, as for most pairs of  $(D_{bid}, I_0)$  we have  $\Delta_r^* \ll 1$ . For Amazon, Cisco, Dell, Ebay and Oracle the optimal display ratio can still be significantly above zero for sufficient opposite-side (i.e. negative), initial imbalance and not too large same-side depth. Apart from Apple and Hewlett-Packard, we observe that generally negative initial imbalances (i.e. opposite-side liquidity excess) lead to higher exposure. This suggests, that increasing already-skewed imbalances on the same side by exposing further shares is generally sub-optimal. On the other hand, when the imbalance is skewed towards the opposite side, then exposure seems to be a proper decision.<sup>36</sup> For large order sizes (right-hand side of figure 8 and 9) the optimal strategy is zero exposure (“hiding the order”). Again taking into account that our markets obey  $\frac{\partial \mu}{\partial I} > 0$ , this is in line with Proposition 2.

Having estimated the price and flow parameters and how they relate to the imbalance, we find that our model confirms many naive expectations. Namely, for larger Iceberg Orders the trader ideally reduces her display ratio, since large orders (if displayed) generally incur large adverse market movements. More precisely and in the sense of market impact elaborated upon in previous sections, for a buy trader, large positive imbalances cause the flows to react adversely, i.e. competing limit order flows increase on the same side and thus increases traders’ *execution risk*. In addition, the expected best ask price process does also moves away, increasing

<sup>33</sup>Previously, Hautsch and Huang have studied the impact of single limit orders on the spot price only Hautsch and Huang (2009).

<sup>34</sup>Optimal exposure strategies for intermediate exposures are colored using a specific rainbow-color-gradient.

<sup>35</sup>See Figure 7.

<sup>36</sup>These statements have to be taken with great caution, since this is not an extensive cross-sectional study.

*opportunity risk*. Hence large imbalances incur market impact in two ways. First they reduce the chance of getting executed by increased liquidity competition on the same-side market and secondly the trader is forced to pay a higher premium for the unexecuted part of the Iceberg order. On the other hand, small orders, since they do not substantially change the imbalance of the book, don't incur substantial market impact. Hence our trader can fully benefit from priority-gain when exposing the whole order. For intermediate-sized orders, depending on the current order book state as well as the order size and the trading interval, there is an optimal display ratio. These observations are in total accordance with the expectation that large orders cause high market impact and thus justify the need for hiding trading intentions, while small orders can be freely disclosed to the market. At the same time, we find that a trader should display more the larger the observable depth at the same side  $D_{bid}$  that sits ahead of him with higher priority-to-execution. Again, we argue that in the case of large queue sizes the additional display of the iceberg order does only have a limited effect on the relative imbalance and thus does not cause much market impact. These results are broadly consistent with the empirical analysis on traders "real" order exposure strategies as reported in Bessembinder et al. (2009). However, as our analysis is conditioned on the (initial) state of the book and thus is more refined as compared to their cross-sectional regression study. According to our model, we find that the optimal exposure strategy is not always consistent with what traders do in general as indicated by their work. Namely, depending on the state of the book and the stock and the time horizon considered, traders should not expose their trade intentions when they observe larger depth or imbalances at the same side.

### 3.5 The Performance Test: Iceberg versus Limit Orders

Besides knowing optimal exposure strategies, in this final sequel, we address the question to what degree the usage of an Iceberg order may improve the trading process as compared to using plain limit orders. This question is highly relevant for market participants and investors.

Let  $W_{ice} := \min_{0 \leq \Delta \leq N} W(\Delta)$  denote the (optimal) Iceberg and  $W_{lim} := W(N)$  the corresponding limit order performance. Then the relative difference between both performances can be written as

$$\sigma := \frac{W_{ice} - W_{lim}}{W_{lim}} \leq 0. \quad (3.3)$$

The smaller  $\sigma$ , the better-off the trader is using an Iceberg Order; for  $\sigma$  close to zero she may as well use plain limit orders. Moreover, it is of interest to know under what market-conditions the usage of Iceberg orders is generally preferred.

The performance for Apple, Amazon, Cisco, Dell, Ebay, Hewlett-Packard, Microsoft and Oracle for  $\Delta t = 10s$  and spreads equaling one cent are shown in the Figures 10 and 11 (see appendix). Figure 10, shows  $\sigma$  as a function of the (initial) depth  $D_{bid}$  and the book-imbalance  $I_0$ ; Figure 11 plots the performance depending on order sizes  $N$  and imbalances  $I_0$ . Green-blue-colored (red-colored) faces represent regions, where Iceberg's do (not) provide additional cost-savings as compared to plain limit orders. We see that - ceteris paribus - the usage of Iceberg Orders is beneficial, when the (initial) imbalance on the trader's side the order size is large. Our interpretation is as follows. Since small orders don't significantly alter the state of the book, such orders won't incur much market impact and thus the cost-savings generated by hiding

shares via Iceberg orders is limited.<sup>37</sup> For large orders however, since the potential to cause large *exposure impact*, the cost-savings can be highly significant. Indeed according to Figure 11, we find throughout all stocks that for a sufficiently large order sizes the performance gain with Icebergs over plain limit orders can reach up to 60%. This result is consistent with earlier empirical results, that Iceberg Orders are prevalently used by large traders.<sup>38</sup> For both figures we observe that an Iceberg Order can boost trading performance particularly when imbalance-excess is massively on the trader’s side. For instance, consider the Dell-stock in Figure 10. While in the case of strong opposite-side imbalance excess, the performance-gain in using Icebergs is at most 10%, the latter can reach up to 50% for sufficiently strong same-side imbalance-excess ( $I > 0.5$ ). This feature is roughly stable among all stocks in the Figures 11 and 10. We observe that the Iceberg performance  $\sigma$  seems to be broadly independent of the (initial) same-side depth  $D_{bid}$ .

In conclusion, depending on the trading regime (order size and order book state), the optimal display decision can significantly improve trading performance. The results indicate that when the the order size is large and the (initial) imbalance is skewed towards the trader’s side of the book, the trader can significantly enhance execution when using Icebergs. In these situations, we report performance gains of up to 60% as compared to the benchmark limit order case. We also report “regimes” where the usage of Iceberg order can be roughly considered obsolete, particularly for large opposite-side imbalance-excess and small order sizes.

## 4 Conclusion

We proposed a simple model of optimal portfolio liquidation using Iceberg-Orders for risk-neutral traders seeking to minimize transaction costs. Market dynamics are driven by sequential, (aggregated) order submissions and cancellations at the trader’s side and a conditionally independent price process for the opposite side of the market. The key assumption is that the state of the orderbook, especially the volume imbalance at the top of the book, affects the market. As a consequence, by choosing the display size, the trader can control how much *exposure impact* she is willing to generate. Exposure impact, that is, market impact of exposed limit orders, materializes in three important ways: via market and limit order flow and quoted prices. Our empirical analysis suggests that exposure impact is mainly governed by the impact of displayed orders on the limit order flow and the quoted best opposite price, while market order flow shows only weak sensibility towards exposure. In particular, consistent with Harris’ *parasitic trader*-scenario, we find that exposure causes higher limit order competition *at* and *ahead* the trader’s submission price level, while prices (on average) move away from the side of exposure. Less pronounced, we observe that exposure causes reduced market order flow volume as *impatient* trader’s retreat from the market as predicted by Moinas (2010). These findings indicate that markets systematically penalize order disclosure. At the same time, there is a loss in time priority associated with non-disclosure of orders. The trader thus faces a trade-off between a loss in time-priority and a reduced market impact when shielding his trading in-

---

<sup>37</sup>But this does not necessarily imply that sufficiently small orders should be fully disposed as can be seen from figure 11. Still, in order to trade optimally, one might need to even (partially) hide small orders.

<sup>38</sup>See Bessembinder et al. (2009); Harris (1996); Frey and Sandas (2009); D’Hondt et al. (2004); Aitken et al. (2001); De Winne and D’Hondt (2007).



tentions from public view. Based on an analytical analysis we find that it is optimal to hide more when the order is large, depths at the top of the book is small, imbalance-excess is on the same-side of the book, liquidity consumption rates (i.e. market order volumes) are high and liquidity competition (i.e. limit order flow in the same side) is low, a behavioral pattern had been documented earlier by Bessembinder et al. (2009). This also implies that traders indeed make sensible (“optimal”) decisions with regard to exposure. Consistent with their finding, that exposure amounts to higher execution volume, while hiding generally reduces transaction costs, we show that when markets penalize exposure traders who try to maximize execution volume expose more than traders who try to minimize execution costs. This is due to the fact, that in contrast to “execution-cost-traders”, “volume-traders” systematically under-estimates “opportunity” costs.

In demonstrating how the model can be applied to address low-time-scales execution problems in electronic limit order books, we calibrate the model and provide optimal exposure profiles for a range of high-tech stocks with respect to the (initial) state-of-the book. The sample calibration results show that throughout the selected stock-range and the parameter space, exposure ratio decreases with order size, i.e. the trader is supposed to display smaller fractions of the total order when the corresponding size is large. For sufficiently small order sizes, irrespective of the market conditions and the state of the book, the trader can generally expose his full trade intentions as *exposure-impact* is negligible. For intermediate order sizes, one observe that exposure may only be a sensible decisions for a certain sub-spaces in the order book state space, namely for opposite-side imbalances and small initial bid depths  $D_{bid}$ .<sup>39</sup> For sufficiently large order sizes, the optimal strategy is to hide the full order. Finally we benchmark the Iceberg performance against the plain limit order. We find that the improvement in execution performance can be highly substantial, reaching up to 60% percent as compared to the limit order. In general, Icebergs enhance trade performance when the (initial) imbalance  $I_0$  is sufficiently skewed towards the trader’s side and the traders order size is large. Empirical literature has reported that hidden liquidity is used much less for liquid stocks.<sup>40</sup> However, the benchmark test for liquid stocks like Cisco or Microsoft shows that under certain order book conditions execution performance-gain with Icebergs can be significant. The order book size and the imbalance play an important role in this context.

Our findings have important implications for investors and traders as we add further theoretical analysis and guidance to the question how to optimally use hidden orders in limit order book markets and how states of the order-book affect these decisions. This is in particular relevant for low-time scales or high-frequency trading as for (liquid) stocks, orders with typical order sizes are filled at second’s or minute’s time. Venue-owners and regulators on the other hand have an interest to understand the motifs on using hidden liquidity in order to fine-tune the optimal degree of market opacity or pre-trade transparency.

---

<sup>39</sup>Exceptions are reported.

<sup>40</sup>See Bessembinder et al. (2009).

## Time of Iceberg

$\mathcal{B}_0$

Figure 4: Order-Book-  
 $\Delta$  shares (red-colored)  
Hidden  $h$  (light blue-colored)  
presence of submission  
hidden part of the Iceberg  
orders orders arrive (size)  
slip in front of the hidden  
total Iceberg and thus

Order Arrival: The Execution Volume

$$\begin{matrix} x < Q^d \\ B_0 \\ V_T = 0 \end{matrix}$$

$$\begin{matrix} Q^d < x < Q^d + \Delta \\ B_0 \\ V_T = x - Q^d \end{matrix}$$

$$\begin{matrix} Q^d < x \leq Q^h \\ B_0 \\ V_T = \Delta \end{matrix}$$

$$\begin{matrix} \Delta + Q^h < x \leq Q^h + N - \Delta \\ B_0 \\ V_T = \Delta + x - Q^h \end{matrix}$$

Figure 5: Illustration of the Iceberg's Execution Volume  $V_T$  according to (2.4). Since the Iceberg's hidden and visible part have different priority-status, they can not get executed in a contiguous way as in the limit order case. Rather, when computing the execution volume - upon arrival of market order - we have to take into account that within the priority-queue, the hidden and visible Iceberg-parts are separated by "alien" liquidity. We thus need to make an explicit case study. In the top-left picture, we show the case when market orders don't fully execute the liquidity ahead the Iceberg (no Iceberg execution). In the top-right picture, some visible Iceberg-part gets executed. In the bottom-left picture, all visible Iceberg shares get executed. Finally in the bottom-right picture, all visible shares plus some hidden Iceberg shares get executed. If the market order volume exceeds executes all liquidity and the Iceberg - i.e.  $x \geq Q^h + N - \Delta = N + h + D_{bid} + \hat{y} + y^-$ , then of course  $V_T = N$  holds.

## A Stock-Properties

	$\Delta t$ ( <i>sec</i> )	<i>price</i>	<i>spread</i> ( <i>ticks</i> )	<i>spread</i> ( <i>bps</i> )	$D_{bid}$ ( <i>TopBook</i> )	$\alpha \cdot p$ ( $E[x]$ )	$\alpha$ ( <i>TradeSize</i> )	$\hat{\beta} \cdot \hat{q}$ ( $E[\hat{y}]$ )	$\alpha p - \hat{\beta} \hat{q}$ $E[x - \hat{y}]$	$\beta \cdot q$ ( $E[y]$ )	$C$ ( <i>CancelTop</i> )	$\frac{D_{bid}}{\alpha}$
AAPL	3	88.81	1.92	2.16	505	640	863	401	239	311	0.53	4.12
	10	88.8	1.92	2.17	501	1952	2024	1174	778	509	0.61	1.23
	30	88.8	1.93	2.17	499	5572	5573	3233	2339	803	0.65	0.15
AMZN	3	53.3	2.91	5.46	324	215	494	153	62	138	0.40	4.03
	10	53.3	2.92	5.48	322	611	805	481	130	233	0.54	2.31
	30	53.3	2.93	5.50	324	1730	1784	1336	394	374	0.61	0.77
CSCO	3	16.68	1.02	6.13	19776	932	2860	242	690	3753	0.18	128.43
	10	16.68	1.03	6.16	19741	2708	4287	2391	317	7267	0.34	89.29
	30	16.68	1.03	6.16	19686	7764	8548	7700	64	12118	0.50	48.35
DELL	3	10.54	1.05	9.93	8520	600	2617	63	537	1185	0.12	79.76
	10	10.54	1.05	9.96	8498	1477	3098	651	826	2377	0.23	63.10
	30	10.54	1.05	9.98	8480	3682	4835	2219	1463	4234	0.39	51.70
EBAY	3	14.32	1.03	7.21	5904	234	925	28	206	1070	0.16	47.96
	10	14.31	1.04	7.25	5824	643	1192	440	203	2016	0.32	33.55
	30	14.31	1.04	7.28	5840	1746	1957	1713	33	3783	0.52	20.64
HPQ	3	36.33	1.27	3.49	945	181	500	150	31	422	0.45	5.14
	10	36.33	1.27	3.50	942	501	756	388	113	641	0.62	3.89
	30	36.33	1.27	3.51	942	1433	1557	951	482	904	0.72	2.00
MSFT	3	19.99	1.03	5.15	22213	956	2902	519	437	3632	0.17	122.58
	10	19.99	1.03	5.17	22216	2716	4338	3141	-425	6904	0.31	89.82
	30	19.98	1.04	5.18	22140	7470	8309	9346	-1876	11819	0.48	46.69
ORCL	3	18.09	1.03	5.69	13022	887	2834	173	714	2452	0.17	76.63
	10	18.09	1.03	5.70	12938	2479	4255	1477	1002	4909	0.32	53.13
	30	18.09	1.03	5.71	12920	6680	7659	5198	1482	8907	0.50	28.82

Table 1: Average Stock Properties for all periods of 3s, 10s and 30s. The table represents unconditional expectation values for price (mid-point price) the spread (ticks as well as in basis-points relative to the actual quoted mid-point-price), the best bid depth or top-of-the-book ( $D_{bid}$ ), the period’s total buy-side trade volume ( $E[x]$ ), the “trade size” ( $E[x|x > 0]$ ), the limit order flow volume “ahead” submission-price-level ( $E[\hat{y}]$ ), the net-flow volume “ahead” submission-price level ( $E[x - \hat{y}]$ ), the limit order flow volume “at” submission-price level ( $E[y]$ ), the top-of-book cancellation ratio ( $C$ ) and finally the ratio between average top of book depth and the average trade size ( $\frac{D_{bid}}{\alpha}$ ).

AAPL(a)
AAPL(b)
AMZN(a)
AMZN(b)
CSCO
DELL
EBAY
HPQ
MSFT
ORCL

Table 2: Correlation between  
 Apple and Amazon are  
 other representing large  
 at one-tick spreads, see  
 0.001, 0.01, and 0.1 lev

Stock	$\Delta t$ (in s)	$1 - q$	$1 - \hat{q}$	$1 - p$	$1 - r$
AAPL	3	0.12	0.44	0.26	0.47
	10	0.05	0.24	0.04	0.47
	30	0.03	0.14	0	0.47
AMZN	3	0.35	0.57	0.57	0.46
	10	0.15	0.33	0.24	0.46
	30	0.07	0.18	0.03	0.46
CSCO	3	0.09	0.97	0.67	0.43
	10	0.02	0.87	0.37	0.43
	30	0.02	0.71	0.09	0.43
DELL	3	0.19	0.98	0.77	0.34
	10	0.04	0.89	0.52	0.34
	30	0.01	0.72	0.24	0.34
EBAY	3	0.23	0.98	0.75	0.49
	10	0.05	0.9	0.46	0.49
	30	0.02	0.73	0.11	0.49
HPQ	3	0.18	0.79	0.64	0.68
	10	0.06	0.61	0.34	0.68
	30	0.03	0.43	0.08	0.68
MSFT	3	0.11	0.96	0.67	0.43
	10	0.02	0.85	0.37	0.43
	30	0.01	0.67	0.1	0.43
ORCL	3	0.12	0.97	0.69	0.49
	10	0.02	0.88	0.42	0.49
	30	0.01	0.72	0.13	0.49

Table 3: Unconditional empirical estimates for the probability that no order arrives in the case of the limit order flows  $\hat{y}$  and  $y$ , market order flow  $x$  and the probability that at arrival-time there, is no hidden liquidity ( $h > 0$ ) at submission-price-level ( $1 - r$ ). Estimates for the flow probabilities have been undertaken for the periods of 3s, 10s and 30s. Due to the nature of the NASDAQ data set, for the “hidden probabilities” we have only one timely-unconditioned estimate. Observe that the chances that no orders arrive decreases with the period length.

## B Estimation-Methodology

The MLE-estimates are constructed as follows. For each realization of the imbalance  $I$ , we denote  $n$  the number of samples of time-length  $\Delta t$  and  $x_i, y_i, \hat{y}_i, h_i, a_i$  the aggregated market and limit order volumes, hidden liquidity volume and (relative) price increment of the  $i$ 'th sample, respectively. Then, the canonical choice for the MLE-estimates of the parameters defined by distribution in (2.9) are given as follows

$$p_I = \frac{\sum_{i=0}^n \mathbb{1}_{\{x_i > 0\}}}{n} \qquad \alpha_I = \frac{\sum_{i=1}^n \mathbb{1}_{\{x_i > 0\}} x_i}{\sum_{i=0}^n \mathbb{1}_{\{x_i > 0\}}} \qquad (\text{B.1})$$

$$q_I = \frac{\sum_{i=0}^n \mathbb{1}_{\{y_i > 0\}}}{n} \qquad \beta_I = \frac{\sum_{i=1}^n \mathbb{1}_{\{y_i > 0\}} y_i}{\sum_{i=0}^n \mathbb{1}_{\{y_i > 0\}}} \qquad (\text{B.2})$$

$$\hat{q}_I = \frac{\sum_{i=0}^n \mathbb{1}_{\{\hat{y}_i > 0\}}}{n} \qquad \hat{\beta}_I = \frac{\sum_{i=1}^n \mathbb{1}_{\{\hat{y}_i > 0\}} \hat{y}_i}{\sum_{i=0}^n \mathbb{1}_{\{\hat{y}_i > 0\}}} \qquad (\text{B.3})$$

$$r_I = \frac{\sum_{i=0}^n \mathbb{1}_{\{h_i > 0\}}}{n} \qquad \gamma_I = \frac{\sum_{i=1}^n \mathbb{1}_{\{h_i > 0\}} h_i}{\sum_{i=0}^n \mathbb{1}_{\{h_i > 0\}}} \qquad (\text{B.4})$$

$$\mu_I = \frac{\sum_{i=0}^n a_i}{n}. \qquad (\text{B.5})$$

## C Estimation Results

PSfrag replacements

DELL  
CSCO  
EBAY  
HPQ  
MSFT  
ORCL  
AMZN  
AAPL

$I$   
 $\bar{E}[\hat{y}|I]$   
 $\bar{E}[y|I]$

PSfrag replacements

~~shares~~

~~Probability~~

~~$\Delta t = 3s$~~

~~$\Delta t = 10s$~~

~~$\Delta t = 30s$~~

ORCL  
AMZN  
AAPL

$I$   
 $\bar{E}[\hat{y}|I]$   
 $\bar{E}[y|I]$

PSfrag replacements

~~shares~~

~~Probability~~

~~$\Delta t = 3s$~~

~~$\Delta t = 10s$~~

~~$\Delta t = 30s$~~

ORCL  
AMZN  
AAPL

$I$   
 $\bar{E}[\hat{y}|I]$   
 $\bar{E}[y|I]$

$\bar{E}[x|I]$

shares

Probability

$\Delta t = 3s$

$\Delta t = 10s$

$\Delta t = 30s$

PSfrag replacements

DELL  
CSCO  
EBAY  
HPQ  
MSFT  
ORCL  
AMZN  
AAPL

$I$   
 $\bar{E}[\hat{y}|I]$   
 $\bar{E}[y|I]$

PSfrag replacements

~~shares~~

~~Probability~~

~~$\Delta t = 3s$~~

~~$\Delta t = 10s$~~

~~$\Delta t = 30s$~~

ORCL  
AMZN  
AAPL

$I$   
 $\bar{E}[\hat{y}|I]$   
 $\bar{E}[y|I]$

PSfrag replacements

~~shares~~

~~Probability~~

~~$\Delta t = 3s$~~

~~$\Delta t = 10s$~~

~~$\Delta t = 30s$~~

ORCL  
AMZN  
AAPL

$I$   
 $\bar{E}[\hat{y}|I]$   
 $\bar{E}[y|I]$

$\bar{E}[x|I]$

shares

Probability

$\Delta t = 3s$

$\Delta t = 10s$

$\Delta t = 30s$

Figure 6: Order Flow Impact of Order Book Imbalance  $I$ . Absolute expected order flow volume -for the three flow types  $x$ ,  $\hat{y}$ ,  $y$ - as multiple of its expected magnitudes when the order book is initially balanced  $I = 0$ , i.e.  $\bar{E}[x|I] := \frac{E[x_I]}{E[\hat{y}|I=0]}$  and analog for  $\hat{y}$  and  $y$ . Order flow reacts stronger on shorter time horizons than on longer. Moreover, the liquidity supplying and price improving  $\hat{y}$ flow reacts most strongly on prevailing order book imbalances.



PSfrag replacements

DELL

CSCO

EBAY

HPQ

MSFT

ORCL

AMZN

AAPL

$I$

$E[A_T | I]$

$\Delta t = 3s$

$\Delta t = 10s$

$\Delta t = 30s$

Figure 7: Price Im

PSfrag replacements	AAPL	AAPL
AAPL	AMZN	AMZN
AMZN	CSCO	CSCO
CSCO	DELL	DELL
DELL	EBAY	EBAY
EBAY	HPQ	HPQ
HPQ	MSFT	MSFT
MSFT	ORCL	ORCL
PSfrag replacements	PSfrag replacements	PSfrag replacements
ORCL	$\Delta_r$	$\Delta_r$
AAPL	AAPL	AAPL
AMZN	AMZN	AMZN
CSCO	CSCO	CSCO
DELL	DELL	DELL
EBAY	EBAY	EBAY
HPQ	HPQ	HPQ
MSFT	MSFT	MSFT
ORCL	ORCL	ORCL
PSfrag replacements	PSfrag replacements	PSfrag replacements
$\Delta_r$	$\Delta_r$	$\Delta_r$
AAPL	AAPL	AAPL
AMZN	AMZN	AMZN
CSCO	CSCO	CSCO
DELL	DELL	DELL
EBAY	EBAY	EBAY
HPQ	HPQ	HPQ
MSFT	MSFT	MSFT
ORCL	ORCL	ORCL
PSfrag replacements	PSfrag replacements	PSfrag replacements
$\Delta_r$	$\Delta_r$	$\Delta_r$
AAPL	AAPL	AAPL
AMZN	AMZN	AMZN
CSCO	CSCO	CSCO
DELL	DELL	DELL
EBAY	EBAY	EBAY
HPQ	HPQ	HPQ
MSFT	MSFT	MSFT
ORCL	ORCL	ORCL
$\Delta_r$	$\Delta_r$	$\Delta_r$
$N =$	$N =$	$N =$
$D_{bid}$	$D_{bid}$	$D_{bid}$
Optimal Display	Optimal Display	Optimal Display
$I_0$	$I_0$	$I_0$
$Q =$	$Q =$	$Q =$
$p$ (bps)	$p$ (bps)	$p$ (bps)
$W$	$W$	$W$

Figure 8: Iceberg's Optimal Display Strategy ( $\Delta_r^*$ ) for Apple, Amazon, Cisco and Dell are shown for the case  $\Delta t = 10s$  and the spread equaling one cent. (Cumulative) Order sizes are given in shares. The optimal display ratio is computed against the initial order book imbalance ( $I_0$ ) and the initial best-bid depth ( $D_{bid}$ ). The left column represents small-order-size situations, mid-columns represents mid-size orders and right-hand column represents large orders. Larger order sizes ( $N$ ) lead to smaller exposure.

PSfrag replacements	PSfrag replacements	PSfrag replacements
AAPL	AAPL	AAPL
AMZN	AMZN	AMZN
CSCO	CSCO	CSCO
DELL	DELL	DELL
EBAY	EBAY	EBAY
HPQ	HPQ	HPQ
MSFT	MSFT	MSFT
ORCL	ORCL	ORCL
$\Delta_r$	$\Delta_r$	$\Delta_r$
AAPL	AAPL	AAPL
AMZN	AMZN	AMZN
$D_{bid}$	$D_{bid}$	$D_{bid}$
$\mathcal{O}_{Display}$	$\mathcal{O}_{Display}$	$\mathcal{O}_{Display}$
DELL	DELL	DELL
EBAY	EBAY	EBAY
HPQ	HPQ	HPQ
$p$ (bps)	$p$ (bps)	$p$ (bps)
MSFT	MSFT	MSFT
ORCL	ORCL	ORCL
$\Delta_r$	$\Delta_r$	$\Delta_r$
AAPL	AAPL	AAPL
AMZN	AMZN	AMZN
$D_{bid}$	$D_{bid}$	$D_{bid}$
$\mathcal{O}_{Display}$	$\mathcal{O}_{Display}$	$\mathcal{O}_{Display}$
DELL	DELL	DELL
EBAY	EBAY	EBAY
HPQ	HPQ	HPQ
$p$ (bps)	$p$ (bps)	$p$ (bps)
MSFT	MSFT	MSFT
ORCL	ORCL	ORCL
$\Delta_r$	$\Delta_r$	$\Delta_r$
AAPL	AAPL	AAPL
AMZN	AMZN	AMZN
$D_{bid}$	$D_{bid}$	$D_{bid}$
$\mathcal{O}_{Display}$	$\mathcal{O}_{Display}$	$\mathcal{O}_{Display}$
DELL	DELL	DELL
EBAY	EBAY	EBAY
HPQ	HPQ	HPQ
$p$ (bps)	$p$ (bps)	$p$ (bps)
MSFT	MSFT	MSFT
ORCL	ORCL	ORCL
$\Delta_r$	$\Delta_r$	$\Delta_r$
$N =$	$N =$	$N =$
$D_{bid}$	$D_{bid}$	$D_{bid}$
Optimal Display	Optimal Display	Optimal Display
$I_0$	$I_0$	$I_0$
$Q =$	$Q =$	$Q =$
$p$ (bps)	$p$ (bps)	$p$ (bps)
$W$	$W$	$W$

Figure 9: Iceberg’s Optimal Display Strategy ( $\Delta_r^*$ ) for Ebay, Hewlett-Packard, Microsoft and Oracle are shown for the case  $\Delta t = 10s$  and the spread equaling one cent. (Cumulative) Order sizes are given in shares. The optimal display ratio is computed against the initial order book imbalance ( $I_0$ ) and the initial best-bid depth ( $D_{bid}$ ). The left column represents small-order-size situations, mid-columns represents mid-size orders and right-hand column represents large orders. Larger order sizes ( $N$ ) lead to smaller exposure.

PSfrag replacements

AAPL  
AMZN  
CSCO  
DELL  
EBAY  
HPQ  
MSFT  
ORCL  
 $\Delta$

PSfrag replacements

$N =$   
~~AAPL~~  
~~AMZN~~  
~~CSCO~~  
~~DELL~~  
~~EBAY~~  
~~HPQ~~  
~~MSFT~~  
ORCL  
 $\Delta$   
 $N =$   
 $I_0$   
more sig-  
nificant  
 $N =$   
 $D_{bid}$   
 $N$   
 $\sigma$   
P.-Gain

Figure 10: Regions of  $\Delta$  for different stock sizes in shares ( $N$ ). For  $\Delta$  Performance-Gain (“P.-Gain”).

PSfrag replacements

AAPL  
AMZN  
CSCO  
DELL  
EBAY  
HPQ  
MSFT  
ORCL  
ORCL

PSfrag replacements

$\Delta$   
 $N=$   
AAPL  
AMZN  
CSCO  
DELL  
EBAY  
HPQ  
P.-Gain  
MSFT  
ORCL  
ORCL  
 $\Delta$   
 $N=$   
 $I_0$   
more sig-  
nificant  
 $D_b =$   
 $N$   
 $\sigma$   
P.-Gain

PS

PS

Figure 11: Regions of depth  $D$  (for shorter Iceberg orders potential

## References

- AITKEN, M. J., H. BERKMAN, AND D. MAK (2001): “The use of undisclosed limit orders on the Australian Stock Exchange,” *Journal of Banking & Finance*, 25, 1589–1603.
- ALFONSI, A., A. FRUTH, AND A. SCHIED (2010): “Optimal Execution Strategies in Limit Order Books with General Shape Functions,” *Quantitative Finance*, 2, 143–157.
- ALFONSI, A. AND A. SCHIED (2010): “Optimal Trade Execution and Absence of Price Manipulations in Limit Order Book Models,” *SSRN eLibrary*.
- ALMGREN, R. (2001): “Optimal Execution of Portfolio Transactions,” *Journal of Risk*, 3, 5–39.
- (2003): “Optimal Execution with Nonlinear Impact Functions and Trading-enhanced risk,” *Applied Mathematical Finance*, 10, 1–18.
- ALMGREN, R. AND N. CHRISS (1999): “Optimal Execution of Portfolio Transactions,” *Journal of Risk*, 3, 5–39.
- ANGEL, J. (1994): “Limit versus Market Order,” Working papers, Georgetown University.
- BEBER, A. AND C. CAGLIO (2005): “Order Submission Strategies and Information: Empirical Evidence from the NYSE,” *SSRN eLibrary*.
- BERTSIMAS, D. AND A. W. LO (1998): “Optimal Control of Execution Costs,” *Journal of Financial Markets*, 1, 1–50.
- BESSEMBINDER, H., M. PANAYIDES, AND K. VENKATAMARAN (2009): “Hidden Liquidity: An Analysis of Order Exposure Strategies in Electronic Stock Markets,” *Journal of Financial Economics*, 94, 361–383.
- BIAIS, B., P. HILLION, AND C. SPATT (1995): “An Empirical Analysis of the Limit Order Book and the Order Flow in the Paris Bourse,” *The Journal of Finance*, 50, 1655–1689.
- BUTI, S. AND B. RINDI (2008): “Hidden Orders and Optimal Submission Strategies in a Dynamic Limit Order Market,” *SSRN eLibrary*.
- CAO, C., O. HANSCH, AND X. WANG (2009): “The information content of an open limit-order book,” *Journal of Futures Markets*, 29, 16–41.
- CHORDIA, T., R. ROLL, AND A. SUBRAHMANYAM (2002): “Order imbalance, liquidity, and market returns,” *Journal of Financial Economics*, 65, 111 – 130.
- COPELAND, T. E. AND D. GALAI (1983): “Information Effects on the Bid-Ask Spread,” *The Journal of Finance*, 38, 1457–1469.
- DE WINNE, R. AND C. D’HONDT (2007): “Hide-and-Seek in the Market: Placing and Detecting Hidden Orders,” *Review of Finance*, 11, 663–692.
- (2009): “Hidden Liquidity in a Pure Order-Driven Market,” *Finance Letters*, forthcoming.
- D’HONDT, C., R. DE WINNE, AND A. FRANCOIS-HEUDE (2004): “Hidden Orders on Euronext: Nothing is Quite as it Seems,” *SSRN eLibrary*.

- DOMOWITZ, I. AND J. WANG (1994): “Auctions as algorithms : Computerized trade execution and price discovery,” *Journal of Economic Dynamics and Control*, 18, 29 – 60.
- DOMOWITZ, I. AND H. YEGERMAN (2005): “The Cost of Algorithmic Trading: A First Look at Comparative Performance,” *Institutional Investor Journal*, 1, 30–40.
- ELLUL, A., C. W. HOLDEN, P. JAIN, AND R. JENNINGS (2003): “Determinants of Order Choice on the New York Stock Exchange,” .
- ESSER, A. AND B. MÖNCH (2007): “The navigation of an iceberg: The optimal use of hidden orders,” *Finance Research Letters*, 4, 68 – 81.
- FOUCAULT, T. (1999): “Order flow composition and trading costs in a dynamic limit order market,” *Journal of Financial Markets*, 2, 99 – 134.
- FOUCAULT, T., O. KADAN, AND E. KANDEL (2005): “Limit Order Book as a Market for Liquidity,” *Review of Financial Studies*, 18, 1171–1217.
- FREY, S. AND P. SANDAS (2009): “The Impact of Iceberg Orders in Limit Order Books,” *SSRN eLibrary*.
- GLOSTEN, L. R. (1994): “Is the Electronic Open Limit Order Book Inevitable?” *The Journal of Finance*, 49, 1127 – 1161.
- GOETTLER, R. L., C. A. PARLOUR, AND U. RAJAN (2005): “Equilibrium in a Dynamic Limit Order Market,” *The Journal of Finance*, 60, 2149–2192.
- GRIFFITHS, MARK D. AND SMITH, B. F., D. ALASDAIR, S. TURNBULL, AND R. W. WHITE (2000): “The costs and determinants of order aggressiveness,” *Journal of Financial Economics*, 6, 65 – 88.
- HARRIS, L. (1996): “Does a Large Minimum Price Variation Encourage Order Exposure?” Working paper series, University of Southern California.
- (1997): “Order Exposure and Parasitic Traders,” Working paper series, University of Southern California.
- (1998): “Optimal Dynamic Order Submission Strategies in Some Stylized Trading Problems,” *Financial Markets, Institutions & Instruments*, 7, 1–76.
- (2003): *Trading & Exchanges. Market Microstructure for Practitioners*, Oxford University Press.
- HARRIS, L. E. AND V. PANCHAPAGESAN (2005): “The information content of the limit order book: evidence from NYSE specialist trading decisions,” *Journal of Financial Markets*, 8, 25 – 67.
- HAUTSCH, N. AND R. HUANG (2009): “The Market Impact of a Limit Order,” Sfb 649 discussion papers, Sonderforschungsbereich 649, Humboldt University, Berlin, Germany.
- HELLSTRÖM, J. AND O. SIMONSEN (2006): “Does the Open Limit Order Book Reveal Information About Short-run Stock Price Movements?” Umea Economic Studies 687, Umea University, Department of Economics.

- HUANG, R. AND H. STOLL (1994): "Market microstructure and stock return predictions," *Review of Financial Studies*, 7, 179–213.
- JOHNSON, B. (2010): *Algorithmic Trading and DMA: An introduction to direct access trading strategies*, 4Myeloma Press.
- MOINAS, S. (2010): "Hidden Limit Orders and Liquidity in Order Driven Markets," IDEI Working Papers 600, Institut d'conomie Industrielle (IDEI), Toulouse.
- OBIZHAEVA, A. A. AND J. WANG (2005): "Optimal Trading Strategy and Supply/Demand Dynamics," *SSRN eLibrary*.
- PARLOUR, C. (1998): "Price dynamics in limit order markets," *Review of Financial Studies*, 11, 789–816.
- PARLOUR, C. A. AND D. SEPPI (2008): "Limit Order Markets: A Survey," *Forthcoming in Handbook of Financial Intermediation and Banking*.
- PARLOUR, C. A. AND D. J. SEPPI (2003): "LiquidityBased Competition for Order Flow," *Review of Financial Studies*, 16, 301–343.
- PASCUAL GASCO, R. AND D. VEREDAS (2008): "What Pieces of Limit Order Book Information Matter in Explaining Order Choice by Patient and Impatient Traders?" *SSRN eLibrary*.
- RANALDO, A. (2004): "Order aggressiveness in limit order book markets," *Journal of Financial Markets*, 7, 53 – 74.
- ROSU, I. (2009): "A Dynamic Model of the Limit Order Book," *Review of Financial Studies*, 22, 4601–4641.
- SEPPI, D. (1997): "Liquidity provision with limit orders and a strategic specialist," *Review of Financial Studies*, 10, 103–150.
- TUTTLE, L. A. (2003): "Hidden Orders, Trading Costs and Information," *SSRN eLibrary*.