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Curve following in illiquid markets

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Abstract In this article the problem of curve following in an illiquid market is addressed. The optimal control is characterised in terms of the solution to a coupled FBSDE involving jumps via the technique of the stochastic maximum principle. Analysing this FBSDE, we further show that there are buy and sell regions. In the case of quadratic penalty functions the FBSDE admits an explicit solution which is determined via the four step scheme. The dependence of the optimal control on the target curve is studied in detail.

Keywords Stochastic maximum principle · Convex analysis · Fully coupled forward backward stochastic differential equations · Trading in illiquid markets

JEL Classification 93E20 · 91G80 · C02 · C61

1 Introduction

In modern financial institutions, due to external regulation as well as client preferences, there are often imposed trading targets which should be followed. These can take the form of a curve giving desired stock holdings over the course of some time horizon, one could think of a day. In an idealised setting one would simply trade so as to stay exactly on the target. Preventing this is the associated costs, thus one has to balance the two conflicting objectives of ensuring minimal deviation from the prespecified target and concurrently minimising trading costs.

In this article we address and solve the above problem using the techniques of stochastic control. In particular, we prove existence and uniqueness of an optimal control and then give

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a characterisation via the stochastic maximum principle. Our optimal system is described by a fully coupled forward backward stochastic differential equation (FBSDE) and in special cases we provide closed form solutions.

The study of trading in illiquid markets is not new and there exists a large body of literature devoted to this. Rather than reviewing everything we instead concentrate on that which is pertinent to our framework. The typical problem is that of how to optimally liquidate a given stock holding and we mention first the important paper of Almgren [1] in which he formulates a model for temporary and permanent market impact costs and goes on to derive explicit solutions to the liquidation problem. This work has become very popular with practitioners as well as forming the basis for subsequent research articles including Almgren [2], Almgren et al. [4], as well as Schied and Schöneborn [31].

The problem of following a target curve has already been studied in the stochastic control literature. When one wants to track a Brownian Motion this is very similar to the monotone follower problem discussed in Bayraktar and Egami [7], Beneš et al. [9] and Karatzas et al. [22] amongst many others. In the finance literature, H. Leland (unpublished) considers a situation where an investor aims to maintain fixed proportions of his wealth in a given selection of stocks, in a market where there are proportional transaction costs. Explicit solutions are derived, however the optimal strategy has a local time component as in Bayraktar and Egami [7], Davis and Norman [14] and Shreve and Soner [32]. This feature is rather undesirable from the point of view of implementation and thus Pliska and Suzuki [28] reformulate the problem in a market with fixed and proportional transaction costs. Using the techniques of impulse control, they compute explicit strategies and this time, due to the presence of fixed costs, there is no local time phenomenon. In addition they calculate some sensitivities. Let us also mention Palczewski and Zabczyk [25] who extend the model of [28] to the multidimensional case when the underlying prices are Markovian.

Our approach has a certain novelty in comparison to the present literature, in particular we characterise controls via the stochastic maximum principle. The motivation for using such techniques seems quite natural for several reasons, firstly our model has a degenerate forward diffusion component and therefore does not satisfy a uniform ellipticity assumption. This means that standard arguments which may imply a smooth solution to the Hamilton Jacobi Bellman (HJB) equation do not apply. Secondly the current algorithms based on viscosity solutions of such equations are not so amenable to the present problem, again due to the lack of uniform ellipticity. However numerical results based on FBSDEs are not limited by such assumptions and thus seem better suited. Finally and perhaps foremost, our interest is not in the value function per se, but primarily in the optimal control, about which one gets more information with the present methods.

Our first major contribution over the articles [25] and [28] is that we allow for the use of passive orders. Passive orders are a key feature of today's markets and as such it is integral that they are included in the modelling framework. The second contribution is the introduction of a target which may be influenced by possibly many stochastic signals. This offers a large degree of flexibility in inputs and allows for a highly complicated target driven by many market phenomena. Relevant applications include tracking the output of an algorithmic trading program, portfolio liquidation and hedging.

The present article also contributes to the stochastic control literature by showing that it is possible to describe very clearly the structure of the problem by analysing probabilistically the corresponding FBSDE rather than the HJB equation via viscosity solution techniques. In particular we provide a financially relevant problem which can be completely solved using the maximum principle.

The outline of the paper is as follows, Sect. 2 derives the model as well as introducing the target functions, stochastic signal and control problem. Section 3 contains our main results, Sects. 4, 5 and 6 discuss the proofs. We consider the quadratic case in Sect. 7 where we solve the controlled system in closed form and give an explicit solution to the portfolio liquidation problem. We finish in Sect. 8 with some counterexamples showing the necessity of one of our assumptions.

2 The control problem

We consider a finite deterministic time T together with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(s) : s \in [0, T]\}, \mathbb{P})$ satisfying the usual conditions of right continuity and completeness.

Assumption 2.1 The filtration is generated by the following three mutually independent processes,

(i) A d-dimensional Brownian Motion W.
(ii) A one-dimensional Poisson process N with intensity λ.
(iii) A compound Poisson process M with compensator $m(d\theta)dt$ where $m(\mathbb{R}^k) < \infty$.

We consider an investor whose stock holdings are governed by the following SDE,

$$dX^u(s) = u_1(s)N(ds) + u_2(s)ds, \tag{2.1}$$

for $s \in [t, T]$ and with $X^u(t) = x$. The control process $u(\cdot)$ is an $\mathbb{R}^2$-valued process and chosen in the following set,

$$\mathcal{U}_t \triangleq \left\{u \in L^2([t, T] \times \Omega) : u_1 \text{ predictably measurable and } u_2 \text{ progressively measurable}\right\}.$$

The interpretation of u is as follows. The investor places a passive order of size u_1, when a jump of N occurs the order is executed and the portfolio adjusts accordingly. For ease of exposition we consider only full liquidation. The component u_2 represents the market order, interpreted here as a rate, as in [1]. The investor can thus take and provide liquidity.

We use the notation $\|u\|_{L^2}$ to denote the $L^2([t, T] \times \Omega)$-norm of a control, where t will be understood from the context. To keep a distinction we use $\|\cdot\|_{\mathbb{R}^n}$ for the Euclidean norm of an n-dimensional vector, while $|\cdot|$ is reserved for real numbers. Inequalities with respect to random variables are assumed to hold a.s.

In addition to the controlled process X^u, there is an uncontrolled n-dimensional vector Z with dynamics given by

$$dZ(s) = \mu(s, Z(s))ds + \sigma(s, Z(s))dW(s) + \int_{\mathbb{R}^k} \gamma\left(s, Z(s-), \theta\right) \tilde{M}(ds, d\theta), \tag{2.2}$$

for $s \in [t, T]$ and with $Z(t) = z$. Observe that we write $\tilde{M}([0, s] \times A) \triangleq M([0, s] \times A) - m(A)s$ for the compensated Poisson martingale; similarly $\tilde{N}(s) \triangleq N(s) - \lambda s$. The functions μ and σ take values from $[t, T] \times \mathbb{R}^n$ and are valued in $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$ respectively, while γ takes values from $[t, T] \times \mathbb{R}^n \times \mathbb{R}^k$ and is valued in $\mathbb{R}^n$. The vector Z denotes a collection of n factors which may influence the costs of trading as well as the target curve to be followed, however is not affected by the trading strategy of the investor.

Let us now introduce the performance functional,

$$J(t,x,z,u) \triangleq \mathbb{E}\left[\int_t^T g(u_2(s), Z(s)) + h\left(X^u(s) - \alpha(s, Z(s))\right) ds \right.$$
$$\left. + f\left(X^u(T) - \alpha(T, Z(T))\right) \middle| X^u(t) = x, Z(t) = z\right]. \tag{2.3}$$

The cost function g captures the liquidity costs of market orders, the mapping α, from $[t,T] \times \mathbb{R}^n$ to $\mathbb{R}$, is the target function and h and f penalise deviation from the target.

The expression above for the penalisation is rather clear, however let us briefly motivate the origin of the term involving g. Trading takes place in a limit order market, which is characterised by a benchmark price D and a collection of other traders' outstanding limit orders. We assume that the process $(D(s))_{t\le s\le T}$ is a nonnegative martingale. At a given instant s, there are limit sell orders available at prices higher than $D(s)$ and limit buy orders at prices lower than $D(s)$. The investor's market buy order is matched with prevailing limit orders and executed at prices higher than $D(s)$; the more volume demanded, the higher the price paid per share. One may think of this as an increasing supply curve as in Çetin et al. [13]. Similarly, market sell orders are executed at prices lower than $D(s)$ and the price per share is decreasing in the volume sold. The investor may also use passive orders, these are placed and fully executed at $D(s)$. A passive order always achieves a better price, however its time of execution is uncertain.

Given a market order u_2, recall here interpreted as a rate, together with the stochastic signal Z, the above considerations lead us to define the asset price as

$$S(s, Z(s), u_2(s)) = D(s) + \tilde{g}\,(u_2(s), Z(s)), \tag{2.4}$$

where $u_2 \mapsto \tilde{g}(u_2, z)$ is increasing and such that $\tilde{g}(0, z) = 0$. The cash flow over the interval $[t, T]$ is given by

$$\begin{aligned} \mathrm{CF}(u) &\triangleq \int_t^T u_2(s) S(s, Z(s), u_2(s)) ds + \int_t^T u_1(s) D(s-) N(ds) \\ &= \int_t^T [u_2(s) D(s) + u_2(s)\tilde{g}\,(u_2(s), Z(s))] ds + \int_t^T u_1(s) D(s-) N(ds), \end{aligned}$$

where we assume all the necessary conditions for the above stochastic integrals to exist. The premium paid due to not being able to trade at the benchmark price, the cost of trading over the interval $[t, T]$, is then given by

$$\mathrm{CF}(u) - \int_t^T u_2(s) D(s) ds - \int_t^T u_1(s) D(s-) N(ds) = \int_t^T u_2(s)\tilde{g}\,(u_2(s), Z(s))\, ds.$$

Defining the cost function g as $g(u_2, z) \triangleq u_2 \tilde{g}(u_2, z)$ gives precisely the term in (2.3).

Remark 2.2 There are two natural interpretations of the passive order. The first would be as an order placed in a dark venue or crossing network, where the underlying level of liquidity is unobservable, see Hendershott and Mendelson [19] and the references therein for further details. Let us also mention P. Kratz and T. Schöneborn (unpublished) who discuss portfolio liquidation in discrete time in the presence of a dark venue, where one finds portfolio dynamics similar to ours. A second interpretation of the passive order is a stylised version of a limit order where placement is only at the benchmark price and there is no time priority and only full execution.

Remark 2.3 Let us briefly compare the present setting with the literature. Without passive orders, our approach is close to Rogers and Singh [30]. In their model, absolute liquidity costs are captured by a convex, nonnegative loss function. If we set $g(u_2, z) = \kappa u_2^2$ for some $\kappa > 0$, we recover the model of [1]. However therein there is an additional permanent price impact, which is undesirable in our setting.

Remark 2.4 The problem formulated here could be considered for random or infinite time horizon T. However for the applications we have in mind there would be a fixed natural nonrandom timescale, e.g. a day when one thinks of trading into/out of a position using the target curve or possibly an hour where that coincides with the next recalibration of model parameters.

We now proceed to the main problem of interest. The corresponding value function is

$$v(t, x, z) \triangleq \inf_{u \in \mathcal{U}_t} J(t, x, z, u).$$

In the sequel we slightly abuse notation and write $J(u) \triangleq J(t, x, z, u)$. The curve following problem is then defined to be

Problem 2.5 Find $\hat{u} \in \mathcal{U}_t$ such that $J(\hat{u}) = \min_{u \in \mathcal{U}_t} J(u)$.

To ensure existence of an optimal control we need some assumptions on the input functions. We remark that here and throughout the constants may be different at each occurrence.

Assumption 2.6 Each function $\psi = f(\cdot), g(\cdot, z), h(\cdot)$ satisfies:

(i) The function ψ is strictly convex, nonnegative, $\mathcal{C}^1$ and normalised in the sense that $\psi(0) = 0$.

(ii) In addition, ψ has at least quadratic growth, i.e. there exists $\varepsilon > 0$ such that $|\psi(x)| \geq \varepsilon |x|^2$ for all $x \in \mathbb{R}$. In the case of g this is supposed to be uniform in z.

(iii) The functions μ, σ and γ are Lipschitz continuous, i.e. there exists a constant c such that for all $z, z' \in \mathbb{R}^n$ and $s \in [t, T]$,

$$\|\mu(s, z) - \mu(s, z')\|^2_{\mathbb{R}^n} + \|\sigma(s, z) - \sigma(s, z')\|^2_{\mathbb{R}^{n \times d}} + \int_{\mathbb{R}^k} \|\gamma(s, z, \theta) - \gamma(s, z', \theta)\|^2_{\mathbb{R}^n} m(d\theta) \leq c\|z - z'\|^2_{\mathbb{R}^n}.$$

In addition, they satisfy

$$\sup_{t \leq s \leq T} \left[\|\mu(s, 0)\|^2_{\mathbb{R}^n} + \|\sigma(s, 0)\|^2_{\mathbb{R}^{n \times d}} + \int_{\mathbb{R}^k} \|\gamma(s, 0, \theta)\|^2_{\mathbb{R}^n} m(d\theta) \right] < \infty.$$

(iv) The target function α has at most polynomial growth in the variable z uniformly in s, i.e. there exist constants $c_\alpha, \eta > 0$ such that for all $z \in \mathbb{R}^n$,

$$\sup_{t \leq s \leq T} |\alpha(s, z)| \leq c_\alpha \left(1 + \|z\|^\eta_{\mathbb{R}^n}\right).$$

(v) The functions f and h have at most polynomial growth.

Remark 2.7 Let us briefly comment on these assumptions. The nonnegativity assumption is motivated by the fact that trading is always costly together with it never being desirable to deviate from the target. Taking f and h normalised is no loss of generality, this may always be achieved by a linear shift of f, h and α. The convexity and quadratic growth condition lead naturally to a convex coercive problem which then admits a unique solution.

Once existence and uniqueness of the optimal control has been established we shall need further assumptions for a characterisation via an FBSDE.

Assumption 2.8 We require the existence of a constant c such that

(i) The derivatives f' and h' have at most linear growth, i.e. for all $x \in \mathbb{R}$

$$\left|f'(x)\right| + \left|h'(x)\right| \leq c(1+|x|).$$

(ii) The cost function g has polynomial style growth, i.e. for all $u_2 \in \mathbb{R}$

$$\left|u_2 g_{u_2}(u_2,\cdot)\right| \leq c(1+g(u_2,\cdot)).$$

(iii) The cost function g satisfies a subadditivity condition, i.e. for all $u_2, w_2 \in \mathbb{R}$

$$g(u_2+w_2,\cdot) \leq c\,(1+g(u_2,\cdot)+g(w_2,\cdot)).$$

(iv) Constant deterministic controls have finite cost, in particular for all $u_2 \in \mathbb{R}$,

$$\mathbb{E}\left[\int_t^T g(u_2, Z(s))ds\right] < \infty.$$

Remark 2.9 We need the linear growth on the derivatives of f and h to ensure that we can solve the adjoint BSDE. In particular this essentially limits us to quadratic penalty functions h and f. For the cost function g, a typical example satisfying the above assumptions would be to set

$$g(u_2, Z) = cu_2 \arctan(u_2) + u_2^2(Z+\varepsilon),$$

for some $\varepsilon > 0$, where Z is a nonnegative mean-reverting jump process. We think of Z as modelling the inverse order book height. The function $u_2 \arctan(u_2)$ represents a smooth approximation to $|u_2|$ and the constant $c > 0$ represents bid ask spread.

In the present setting, we are most interested in processes on $[t,T]\times\Omega$ and write that a given property $(\mathcal{P})$ holds "$ds \times d\mathbb{P}$ a.e. on B" for a (measurable) subset $B \subset [t,T]\times\Omega$ when $(\mathcal{P})$ holds for the restriction of the measure $ds \times d\mathbb{P}$ to B.

3 Main results

Having formulated the problem and introduced the necessary assumptions, we can now give our main results.

Theorem 3.1 *The functional $u \mapsto J(u)$ is strictly convex for $u \in \mathcal{U}_t$. If Assumption 2.6 holds then for any initial triple $(t,x,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^n$ there is an optimal control, unique $ds \times d\mathbb{P}$ a.e. on $[t,T]\times\Omega$.*

We postpone the proof to Sect. 4. To characterise the optimal control $\hat{u}$ and the corresponding state process $\hat{X} \triangleq X^{\hat{u}}$ we define the following BSDE, the adjoint equation,

$$\begin{aligned} dP(s) &= h'\left(\hat{X}(s) - \alpha(s, Z(s))\right) ds + Q(s)dW(s) + R_1(s)\tilde{N}(ds) \\ &\quad + \int_{\mathbb{R}^k} R_2(s,\theta)\tilde{M}(ds, d\theta), \qquad (3.1) \\ P(T) &= -f'\left(\hat{X}(T) - \alpha(T, Z(T))\right). \end{aligned}$$

Theorem 3.2 *Let Assumptions 2.6 and 2.8 hold. Then*

(i) *The above BSDE has a unique solution for all starting triples* $(t, x, z) \in [0, T]\times\mathbb{R}\times\mathbb{R}^n$.
(ii) *A control* $\hat{u}$ *is optimal if and only if* $ds \times d\mathbb{P}$ *a.e. on* $[t, T] \times \Omega$,

  (a) $\hat{u}_2(s, \omega)$ *is the pointwise minimiser of* $u_2 \mapsto g(u_2, Z(s, \omega)) - P(s, \omega)u_2$.
  (b) $P(s-, \omega) + R_1(s, \omega) = 0$.

Remark 3.3 Theorem 3.2 is essentially a version of the stochastic maximum principle and we now describe how this relates to those in the literature. Our results are most similar to Cadenillas [11], however in his setting one requires that (in our notation) the joint process $Y \triangleq (X, Z)^\intercal$ have dynamics which are jointly affine as functions of the vector Y and control u. This is not necessarily the case for only Lipschitz μ, σ and γ, so that we are outside the scope of the results therein.

The article of Tang and Li [33] considers the case where the dynamics of Y need not be affine, as in the present article, however they require that the control satisfies the following integrability condition

$$\sup_{t\le s\le T} \mathbb{E}\left[\|u(s)\|^8_{\mathbb{R}^2}\right] < \infty,$$

which excludes the L^2-framework considered here. Finally we mention Ji and Zhou [20], where the authors allow for square integrable controls and non-affine dynamics but have no jumps, so that again their results do not cover the present situation.

The proof of item (ii) relies on the stochastic maximum principle and is dealt with in Sect. 5. The above characterisation is still rather implicit, we can describe $\hat{u}_1$ more precisely and for this require the following definition.

Definition 3.4 The *cost-adjusted target function* $\tilde{\alpha}$ is defined to be the pointwise minimiser (with respect to x) of the value function, $\tilde{\alpha}(t, z) \triangleq \arg\min_{x\in\mathbb{R}} v(t, x, z)$.

The fact that $\tilde{\alpha}$ is well defined is a consequence of the convexity of v as well as Lemma 4.2 where it is shown that v has at least quadratic growth in x. The next theorem shows that trading is directed towards the cost-adjusted target function, motivating its definition.

Theorem 3.5 *Let Assumptions 2.6 and 2.8 hold, then*

(i) *The optimal passive order is given* $ds \times d\mathbb{P}$ *a.e. on* $[t, T] \times \Omega$ *by*

$$\hat{u}_1(s, \omega) = \tilde{\alpha}(s, Z(s-, \omega)) - \hat{X}(s-, \omega).$$

(ii) *Define the buy region via*

$$\mathcal{R}_{\text{buy}} \triangleq \left\{(s, x, z) \in [t, T] \times \mathbb{R} \times \mathbb{R}^n : x < \tilde{\alpha}(s, z)\right\},$$

as well as the set where the optimal state process is valued in $\mathcal{R}_{\text{buy}}$,

$$A_{\text{buy}} = \left\{(s, \omega) \in [t, T] \times \Omega : (s, \hat{X}(s-, \omega), Z(s-, \omega)) \in \mathcal{R}_{\text{buy}}\right\}.$$

Then we have that $\hat{u}_1, \hat{u}_2 > 0$, $ds \times d\mathbb{P}$ *a.e. on* A_{buy}. *The symmetric result holds for the sell region,* $\mathcal{R}_{\text{sell}}$, *defined similarly with* $>$ *replacing* $<$.

(iii) *For the corresponding boundary sets,*

$$\mathcal{R}_{\text{no trade}} \triangleq \left\{(s, x, z) \in [t, T] \times \mathbb{R} \times \mathbb{R}^n : x = \tilde{\alpha}(s, z)\right\},$$
$$A_{\text{no trade}} \triangleq \left\{(s, \omega) \in [t, T] \times \Omega : (s, \hat{X}(s-, \omega), Z(s-, \omega)) \in \mathcal{R}_{\text{no trade}}\right\},$$

we have $\hat{u}_1 = \hat{u}_2 = 0$, $ds \times d\mathbb{P}$ *a.e. on* $A_{\text{no trade}}$.

The proof of this result and a discussion of further properties of the cost-adjusted target function are given in Sect. 6.

4 Existence of a solution

The aim of this section is to establish existence of an optimal control. This is done in several steps, first some a priori estimates on the growth of the value function are established. These are then used to show that it is sufficient to consider a subset of controls with a uniform L^2-norm bound. This then permits the use of a Komlós argument to construct the optimal control.

We begin with some estimates from the theory of SDEs.

Lemma 4.1 *Let* X^u *and* Z *have dynamics* (2.1) *and* (2.2) *respectively.*

(i) *For every* $p \geq 2$ *there exists a constant* c_p *such that for every* $t \in [0, T]$ *we have*

$$\mathbb{E}\left[\sup_{t\leq s\leq T} \|Z(s)\|^p_{\mathbb{R}^n} | Z(t) = z\right] \leq c_p \left(1 + \|z\|^p_{\mathbb{R}^n}\right).$$

(ii) *There exists a constant* c_x *such that for any* $u \in \mathcal{U}_t$ *we have*

$$\mathbb{E}\left[\sup_{t\leq s\leq T} |X^u(s)|^2 \, | X^u(t) = x\right] \leq c_x \left(1 + \|u\|^2_{L^2}\right).$$

In particular, X^u *has square integrable supremum for all* $u \in \mathcal{U}_t$.

Proof Item (i) is a well known estimate on the solution of an SDE with Lipschitz coefficients, see for example Barles et al. [6, Proposition 1.1]. Item (ii) can be proved by applying standard estimates to the known form of X^u. Note that the constant c_x is uniform in the controls, however depends on x. □

Since the cost and penalty functions have quadratic growth in x and the SDE for X is linear in the control u, it is sensible, at least intuitively, that controls with large L^2-norm cannot be optimal. To verify this mathematically we need some growth estimates on the value function.

Lemma 4.2 *There exist constants* $c_0, c_1, \eta > 0$ *such that*

$$v(t, x, z) \geq c_0 x^2 - c_1 \left(1 + \|z\|^\eta_{\mathbb{R}^n}\right),$$

for all $(t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$.

Proof To ease notation we write the expectation in (2.3) as $\mathbb{E}_{t,x,z}[\cdot]$. Using the quadratic growth of f, g, h yields

$$v(t,x,z) \geq \varepsilon \inf_{u\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\left[\int_t^T |u_2(s)|^2 + \big(X^u(s)-\alpha(s,Z(s))\big)^2 ds + \big(X^u(T)-\alpha(T,Z(T))\big)^2\right].$$

Next an application of the inequality $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ provides

$$v(t,x,z) \geq \frac{\varepsilon}{2}\inf_{u\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\left[\int_t^T |u_2(s)|^2 + \big|X^u(s)\big|^2 ds + \big|X^u(T)\big|^2\right] - \varepsilon\mathbb{E}_{t,x,z}\left[\int_t^T |\alpha(s,Z(s))|^2 ds + |\alpha(T,Z(T))|^2\right].$$

The polynomial growth of α coupled with Lemma 4.1 allows us to write

$$v(t,x,z) \geq \tfrac{\varepsilon}{2}\inf_{u\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\left[\int_t^T |u_2(s)|^2 + \big|X^u(s)\big|^2 ds + \big|X^u(T)\big|^2\right] - c_1\left(1+\|z\|^{\eta}_{\mathbb{R}^n}\right).$$

It now remains only to estimate the infimum. This term may be interpreted as a stochastic control problem with quadratic penalty and cost functions and zero target. It is known that such a control problem admits an analytic solution via Riccati equations. In particular we have that

$$v(t,x,z) \geq \frac{\varepsilon}{2}a(t)x^2 - c_1\left(1+\|z\|^{\eta}_{\mathbb{R}^n}\right),$$

for a function a given by the solution of the differential equation

$$a'(s) = a^2(s) - 1 + \lambda a(s), \quad s\in[t,T], \quad a(T)=1.$$

Solving explicitly for a one finds that it is monotone and that $a(t) > 0$. If we set

$$c_0 \triangleq \frac{\varepsilon}{2}\min\{a(t), a(T)\} > 0,$$

this completes the proof. □

With the above estimate we show that it is sufficient to consider J on a weakly compact subset of $\mathcal{U}_t$. We note here that from Assumption 2.6 together with Lemma 4.1 we deduce that $J(0) < \infty$. This fact is used in the following lemma.

Lemma 4.3 *There is a constant $c_{\max}$ such that $\|u\|^2_{L^2} \geq c_{\max}$ implies that u cannot be optimal.*

Proof For a control $u \in \mathcal{U}_t$ we want to show that we may bound $J(u)$ from below in terms of $\|u\|^2_{L^2}$. For the market order u_2 we have

$$J(u) \geq \mathbb{E}\left[\int_t^T g(u_2(s), Z(s))ds\right] \geq \varepsilon\mathbb{E}\left[\int_t^T |u_2(s)|^2 ds\right], \tag{4.1}$$

where we have used Assumption 2.6 (ii).

The estimate in terms of the passive order u_1 is slightly more involved. Let τ_1 denote the first jump time of the Poisson process N after t, an exponentially distributed random variable with parameter λ, and set $\tau \triangleq \tau_1 \wedge T$. The functions f, g and h are nonnegative, combining this with the definition of v as an infimum we derive

$$\begin{aligned} J(u) &= \mathbb{E}_{t,x,z}\left[\int_t^\tau g(u_2(s), Z(s)) + h\left(X^u(s) - \alpha(s, Z(s))\right) ds\right] \\ &\quad + \mathbb{E}_{t,x,z}\left[J(\tau, X^u(\tau), Z(\tau), u)\right] \\ &\geq \mathbb{E}_{t,x,z}\left[v(\tau, X^u(\tau), Z(\tau))\right], \end{aligned}$$

where J in the above is evaluated at controls on the stochastic interval $[\tau, T]$. Noting the nonnegativity of v this implies the lower bound

$$J(u) \geq \mathbb{E}_{t,x,z}\left[\mathbb{1}_{\{\tau_1 < T\}} v(\tau_1, X^u(\tau_1), Z(\tau_1))\right].$$

Applying first the growth estimates from Lemma 4.2, then combining the inequality

$$\mathbb{1}_{\{\tau_1 < T\}} \|Z(\tau_1)\|^\eta_{\mathbb{R}^n} \leq \sup_{t \leq s \leq T} \|Z(s)\|^\eta_{\mathbb{R}^n},$$

with Lemma 4.1 provides the existence of a constant $c_{1,z}$ such that

$$J(u) \geq c_{1,z} + c_0 \mathbb{E}_{t,x,z}\left[\mathbb{1}_{\{\tau_1 < T\}} |X^u(\tau_1)|^2\right],$$

where $c_0 > 0$ is as in Lemma 4.2. We may write $X^u(\tau_1) = X^u(\tau_1-) + u_1(\tau_1)$ and observe that on the set $\{\tau_1 < T\}$ we have the relation

$$X^u(\tau_1-) = x + \int_t^{\tau_1} u_2(s) ds.$$

Using the inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$ twice, together with the Jensen inequality, we get

$$J(u) \geq c_{1,x,z} + c_0 \mathbb{E}\left[\mathbb{1}_{\{\tau_1 < T\}} |u_1(\tau_1)|^2\right] - c_2 \mathbb{E}\left[\int_t^{\tau_1} |u_2(s)|^2 ds\right],$$

for some constant $c_2 > 0$, where we drop the subscript $\{t, x, z\}$. In light of inequality (4.1) we derive

$$\left(1 + \frac{c_2}{\varepsilon}\right) J(u) \geq c_{1,x,z} + c_0 \mathbb{E}\left[\mathbb{1}_{\{\tau_1 < T\}} |u_1(\tau_1)|^2\right].$$

An application of the law of total expectation and relabelling the constants provides the estimate

$$J(u) \geq c_{1,x,z} + c_0 \int_t^T \lambda \mathbb{E}\left[|u_1(s)|^2\right] e^{-\lambda(s-t)} ds.$$

We apply the uniform bound $e^{-\lambda(s-t)} \geq e^{-\lambda(T-t)}$ for $s \in [t, T]$ in the above, then combine with (4.1) to see that

$$J(u) \geq c_{1,x,z} + c_0 \|u\|^2_{L^2}.$$

In particular if

$$\|u\|_{L^2}^2 \geq c_{\max} \triangleq \frac{J(0) - c_{1,x,z}}{c_0} + 1,$$

then we see that $J(u) > J(0)$ and the control u is clearly not optimal. □

Before completing the proof, we recall a definition and refer the reader to Protter [29] for further details.

Definition 4.4 A sequence of processes $(Y^n)_{n\in\mathbb{N}}$ defined on $[t, T] \times \Omega$ and valued in $\mathbb{R}$ converges to a process $Y : [t, T] \times \Omega \mapsto \mathbb{R}$ *uniformly on compacts in probability (UCP)* if, for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\leq s\leq T} \left|Y_s^n - Y_s\right| > \varepsilon \right) = 0.$$

We now finish the proof of our first main result.

Proof of Theorem 3.1 The strict convexity of J is a direct consequence of the strict convexity of f, g and h. From the previous lemma it follows that (using the notation therein) if we set

$$\mathcal{U}_t^{c_{\max}} \triangleq \left\{u \in \mathcal{U}_t : \|u\|_{L^2}^2 \leq c_{\max}\right\},$$

then

$$\inf_{u\in\mathcal{U}_t} J(u) = \inf_{u\in\mathcal{U}_t^{c_{\max}}} J(u).$$

We take a sequence of minimising processes $(u^n)_{n\in\mathbb{N}} \subset \mathcal{U}_t^{c_{\max}}$. Due to the uniform bound on the L^2-norms we may proceed as in Beneš et al. [10, Theorem 2.1] to find a subsequence (also indexed by n) together with a process $\hat{u} : [t, T] \times \Omega \to \mathbb{R}^2$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} u^j = \hat{u}$$

$ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$. To be precise, we note here that the superscripts index the sequence and the subscripts the components of the process. Due to Karatzas and Shreve [21, Proposition 1.2] we may assume that $\hat{u}_2$ is progressively measurable, whereas the predictability of $\hat{u}_1$ follows as in Applebaum [5, Lemma 4.1.3]. In particular we deduce first the appropriate measurability of $\hat{u} \in \mathcal{U}_t$ and then from Fatou's lemma that $\hat{u} \in \mathcal{U}_t^{c_{\max}}$. Before proving the optimality of $\hat{u}$ we first show some convergence results. For $n \in \mathbb{N}$ we set

$$\bar{u}^n \triangleq \frac{1}{n} \sum_{j=1}^{n} u^j \quad \text{and} \quad \bar{X}^n \triangleq \frac{1}{n} \sum_{j=1}^{n} X^{u^j}.$$

We have the following estimate,

$$\mathbb{E}\left[\sup_{t\leq s\leq T} |\bar{X}^n(s) - \bar{X}^m(s)| \right] \leq \mathbb{E}\left[\int_t^T |\bar{u}_2^n(s) - \bar{u}_2^m(s)| ds \right] + \mathbb{E}\left[\int_t^T |\bar{u}_1^n(s) - \bar{u}_1^m(s)| N(ds) \right].$$

Via the de-la-Vallée-Poussin Theorem, a consequence of the uniform bound on the L^2-norms is that $(\bar{u}^n)_{n\in\mathbb{N}}$ also converges in $L^1([t, T] \times \Omega)$ to $\hat{u}$. Using Definition 4.4 together with the Markov inequality it now follows that $(\bar{X}^n)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{D}$, the space of càdlàg processes equipped with the UCP topology. From [29, Chap. 2] it is known that this space is complete and hence there exists a process $\hat{X}$ such that $\bar{X}^n$ converges to $\hat{X}$. Note that from the above equation it actually follows that we have the stronger convergence

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\le s\le T} |\bar{X}^n(s) - \hat{X}(s)|\right] = 0.$$

In particular the above argument implies that $\hat{X} = X^{\hat{u}}$ up to indistinguishability and is well defined. For the optimality, applying Fatou's lemma together with the convexity of f, g and h gives

$$J(\hat{u}) \le \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} J(u^j) = \inf_{u\in\mathcal{U}_t} J(u).$$

Turning to uniqueness, suppose $\hat{u}$ and $\bar{u}$ are optimal controls. The strict convexity of $u_2 \mapsto g(u_2, z)$ implies that $\hat{u}_2 = \bar{u}_2\ ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$. We also have $X^{\hat{u}} = X^{\bar{u}} ds \times d\mathbb{P}$ a.e. since otherwise $J\left(\frac{\hat{u}+\bar{u}}{2}\right) < J(\hat{u}) = J(\bar{u})$ due to the strict convexity of $x \mapsto h(x)$. An application of Lemma A.1 now provides $\bar{u} = \hat{u} ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$. □

5 The stochastic maximum principle

In this section we are concerned with the proof of Theorem 3.2. In particular we show the existence of a solution to the adjoint equation as well as providing a characterisation of the optimal control. To avoid difficulties related to controls for which the performance functional is not finite we define a subclass of controls given by

$$\mathcal{U}_{\text{adm}} \triangleq \{u \in \mathcal{U}_t : J(u) < \infty\}.$$

It is clear that $\hat{u} \in \mathcal{U}_{\text{adm}}$, that minimising J over $\mathcal{U}_{\text{adm}}$ is equivalent to minimising over $\mathcal{U}_t$ and that for any $u \in \mathcal{U}_{\text{adm}}$ we have

$$\mathbb{E}\left[\int_t^T g(u_2(s), Z(s))ds\right] < \infty.$$

We shall use these properties throughout.

We begin by recalling the adjoint BSDE on $[t, T]$ given in Eq. 3.1,

$$\begin{aligned} dP(s) &= h'\left(\hat{X}(s) - \alpha(s, Z(s))\right) ds + Q(s)dW(s) + R_1(s)\tilde{N}(ds) \\ &\quad + \int_{\mathbb{R}^k} R_2(s,\theta)\tilde{M}(ds, d\theta), \\ P(T) &= -f'\left(\hat{X}(T) - \alpha(T, Z(T))\right). \end{aligned}$$

Remark 5.1 As before Q and R are $\mathbb{R}^d$ and $\mathbb{R}^2$-valued respectively. Strictly speaking the adjoint equation should be for a vector $(P_1, \ldots, P_{n+1})$, where P_1 satisfies the BSDE above. However when one writes down the full system one sees that, due to the fact that the control does not enter the signal Z, P_1 and $(P_2, \ldots, P_{n+1})$ are decoupled. Moreover, we shall see that the optimality criterion only involves P_1, thus it is sufficient to omit $(P_2, \ldots, P_{n+1})$ and to consider only the above BSDE.

Due to the simple structure of the adjoint equation, the proof of existence and uniqueness of a solution is straightforward and we omit it, see for example [33, Lemma 2.4].

Lemma 5.2 *There is a solution* (P, Q, R) *to the adjoint BSDE such that*

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|P(s)|^2+\int_t^T\|Q(s)\|^2_{\mathbb{R}^d}ds+\int_t^T|R_1(s)|^2ds+\int_t^T\int_{\mathbb{R}^k}|R_2(s,\theta)|^2m(d\theta)ds\right]<\infty.$$

It is unique amongst triples (P, Q, R) *satisfying the above integrability criterion.*

The stochastic maximum principle exploits the convexity of the performance functional J together with the fact that the minimum of J can be characterised using the subgradient inequality, which allows us to give an explicit condition for the optimality of a control $\hat{u}$. The following lemma provides the explicit formula for the Gâteaux derivative. We give a proof based upon Cadenillas and Karatzas [12, Lemma 1.1], however we avoid the measurable selection argument therein. In this section the starting point (t, x, z) is fixed and we therefore write $\mathbb{E}[\cdot]$ for $\mathbb{E}_{t,x,z}[\cdot]$.

Lemma 5.3 *For* $u, w \in \mathcal{U}_{\text{adm}}$ *the Gâteaux derivative of* J *is given by*

$$\begin{aligned}\langle J'(w),u\rangle &= \mathbb{E}\left[\int_t^T\left(X^u(s)-x\right)h'\left(X^w(s)-\alpha(s,Z(s))\right)ds\right]\\ &\quad+\mathbb{E}\left[\int_t^T u_2(s)g_{u_2}\left(w_2(s),Z(s)\right)ds\right]\\ &\quad+\mathbb{E}\left[\left(X^u(T)-x\right)f'\left(X^w(T)-\alpha(T,Z(T))\right)\right].\end{aligned}$$

Proof We first note that for $s \in [t, T]$, $\rho \in [0, 1]$ and $u, w \in \mathcal{U}_{\text{adm}}$ we have

$$\begin{aligned}X^{w+\rho u}(s) &= x+\int_t^s(w_2(r)+\rho u_2(r))\,dr+\int_t^s(w_1(r)+\rho u_1(r))\,N(dr)\\ &= X^w(s)+\rho(X^u(s)-x).\end{aligned}$$

Using the fact that the signal is unaffected by the control together with the mean value theorem we may compute, for $\rho \in [0, 1]$,

$$\begin{aligned}\langle J'(w), u\rangle &= \lim_{\rho\to 0}\frac{J(w+\rho u)-J(w)}{\rho}\\ &= \lim_{\rho\to 0}\Bigg\{\mathbb{E}\left[\int_t^T\int_0^1 (X^u(s)-x)h'(X^w(s)+\zeta\rho(X^u(s)-x)-\alpha(s,Z(s)))d\zeta ds\right]\\ &\quad+\mathbb{E}\left[\int_t^T\int_0^1 u_2(s)g_{u_2}\left(w_2(s)+\zeta\rho u_2(s), Z(s)\right)d\zeta ds\right]\\ &\quad+\mathbb{E}\left[\int_0^1 \left(X^u(T)-x\right)f'\left(X^w(T)+\zeta\rho\left(X^u(T)-x\right)-\alpha(T,Z(T))\right)d\zeta\right]\Bigg\}.\end{aligned}$$

Due to the convexity of the functions f, $g(\cdot, z)$ and h, exactly as in [12, Lemma 1.1], the integrands are all decreasing as ρ decreases so that all limits are well defined as $\rho \to 0$. The statement of the lemma will follow from the monotone convergence theorem once we show that

$$\begin{aligned}&\mathbb{E}\left[\int_t^T \left(X^u(s)-x\right)h'\left(X^w(s)+\left(X^u(s)-x\right)-\alpha(s,Z(s))\right)ds\right]\\ &\quad+\mathbb{E}\left[\int_t^T u_2(s)g_{u_2}\left(w_2(s)+u_2(s), Z(s)\right)ds\right]\\ &\quad+\mathbb{E}\left[\left(X^u(T)-x\right)f'\left(X^w(T)+\left(X^u(T)-x\right)-\alpha(T,Z(T))\right)\right]<\infty. \qquad (5.1)\end{aligned}$$

Using the linear growth of h' together with the Young inequality one can find a constant c such that for $s \in [t, T]$

$$\begin{aligned}&\left(X^u(s)-x\right)h'(X^w(s)+X^u(s)-\alpha(s,Z(s)))\\ &\quad\le c\left(1+|X^u(s)|^2+|X^w(s)|^2+|\alpha(s,Z(s))|^2\right).\end{aligned}$$

The right hand side is integrable over $[t, T]\times\Omega$ thanks to Lemma 4.1. An identical argument may be applied to the term involving f'. Thus to complete the proof we need an estimate for the second term in (5.1). The subgradient inequality together with the nonnegativity of g implies that for $s \in [t, T]$ the following holds,

$$\begin{aligned}&u_2(s)g_{u_2}(w_2(s)+u_2(s), Z(s))\\ &\quad\le g\left(u_2(s), Z(s)\right)+\left(w_2(s)+u_2(s)\right)g_{u_2}\left(w_2(s)+u_2(s), Z(s)\right).\end{aligned}$$

Applying now Assumption 2.8(ii) and (iii) and observing that $u, w \in \mathcal{U}_{\text{adm}}$ completes the proof. □

Having constructed a formula for the Gâteaux derivative we may now turn to characterising the optimal control. From the integration by parts formula we derive, for a control $u \in \mathcal{U}_{\text{adm}}$ and $s \in [t, T]$,

$$
\begin{aligned}
&P(s)X^u(s) - P(t)X^u(t) - \int_t^s \psi(r, u(r))dr \\
&\quad = \int_t^s X^u(r-)Q(r)dW(r) + \int_t^s \left[(P(r-) + R_1(r))\, u_1(r) + X^u(r-)R_1(r)\right] \tilde{N}(dr) \\
&\qquad + \int_t^s \int_{\mathbb{R}^k} X^u(r-)R_2(r, \theta)\tilde{M}(dr, d\theta),
\end{aligned}
$$

where we have used that $[N, M] = 0$, a consequence of the independence of M and N together with [5, Proposition 1.3.12], as well as setting

$$
\psi(r, u(r)) \triangleq P(r)u_2(r) + \lambda u_1(r)[P(r-) + R_1(r)] + X^u(r)h'\left(\hat{X}(r) - \alpha(r, Z(r))\right).
$$

We rewrite the above as

$$
Y^u(s) = P(t)X^u(t) + L^u(s), \tag{5.2}
$$

where the processes Y^u and L^u are defined for $s \in [t, T]$ by

$$
\begin{aligned}
Y^u(s) &\triangleq P(s)X^u(s) - \int_t^s \psi(r, u(r))dr, \\
L^u(s) &\triangleq \int_t^s X^u(r-)Q(r)dW(r) + \int_t^s [(P(r-) + R_1(r))\, u_1(r) + X^u(r-)R_1(r)]\tilde{N}(dr) \\
&\quad + \int_t^s \int_{\mathbb{R}^k} X^u(r-)R_2(r, \theta)\tilde{M}(dr, d\theta).
\end{aligned}
$$

Lemma 5.4 *For all $u \in \mathcal{U}_{\text{adm}}$ the process L^u is a martingale.*

Proof For the continuous local martingale part, the result follows from the Burkholder–Davis–Gundy and Hölder inequalities noting that X^u has square integrable supremum and $Q \in L^2([t, T] \times \Omega)$. For each of the remaining terms we may apply Lemma A.3, where the appropriate integrability follows from the fact that X^u and P have square integrable suprema and Q, R_1 and R_2, as well as u, are square integrable. □

We can now turn to the proof of our second main result, the stochastic maximum principle. It is based on ideas given in [11] and extended to the case where the dynamics are not affine in $Y \triangleq (X, Z)^\mathsf{T}$, as discussed in Remark 3.3. We note however that our theorem is not a straightforward extension thereof. We must verify firstly that [11, Assumptions 3.1 and 4.1] hold in this new setting and secondly that one can still derive an a priori L^2-estimate on the control for our choice of cost functions and Z dynamics. This motivates the present detailed derivation of the maximum principle.

Proof of Theorem 3.2 We are minimising a convex functional over $\mathcal{U}_{\text{adm}}$, so by Ekeland and Témam [15, Proposition 2.2.1] a necessary and sufficient condition for optimality of $\hat{u}$ is that

$$
\langle J'(\hat{u}), u - \hat{u}\rangle \geq 0 \quad \text{for all } u \in \mathcal{U}_{\text{adm}}. \tag{5.3}
$$

Due to Lemma 5.4 we know that L^u is a martingale for all $u \in \mathcal{U}_{\text{adm}}$. In particular from Eq. 5.2 we have that $\mathbb{E}[Y^u(T) - Y^{\hat{u}}(T)] = 0$. The definition of Y^u together with the terminal condition in (3.1) allows us to write this as

$$0 = \mathbb{E}\left[P(T)\left[X^u(T) - \hat{X}(T)\right] - \int_t^T \left[\psi(s, u(s)) - \psi(s, \hat{u}(s))\right] ds \right]$$

$$= \mathbb{E}\left[-f'\left(\hat{X}(T) - \alpha(T, Z(T))\right)\left[X^u(T) - \hat{X}(T)\right] - \int_t^T \left[\psi(s, u(s)) - \psi(s, \hat{u}(s))\right] ds \right].$$

Using the known form of the Gâteaux derivative we derive

$$\begin{aligned} \langle J'(\hat{u}), u - \hat{u} \rangle &= \langle J'(\hat{u}), u - \hat{u} \rangle + \mathbb{E}\left[Y^u(T) - Y^{\hat{u}}(T)\right] \\ &= \mathbb{E}\left[\int_t^T \left[u_2(s) - \hat{u}_2(s)\right]\left[g_{u_2}\left(\hat{u}_2(s), Z(s)\right) - P(s)\right] ds \right] \\ &\quad -\lambda \mathbb{E}\left[\int_t^T \left[u_1(s) - \hat{u}_1(s)\right](P(s-) + R_1(s)) ds \right], \end{aligned} \tag{5.4}$$

for every control $u \in \mathcal{U}_{\text{adm}}$.

A consequence of (5.3) and (5.4) is that $\hat{u}$ is optimal if and only if we have $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$,

(i) $\left[u_2(s) - \hat{u}_2(s)\right]\left[g_{u_2}\left(\hat{u}_2(s), Z(s)\right) - P(s)\right] \geq 0 \quad \forall u = (u_1, u_2) \in \mathcal{U}_{\text{adm}}$,
(ii) $P(s-) + R_1(s) = 0$.

To complete the proof of Theorem 3.2, we need to show that we can insert arbitrary $u_2 \in \mathbb{R}$ into (i) and then use the fact that we have the subgradient condition for the pointwise minimum of the strictly convex function

$$u_2 \mapsto g(u_2, Z(s, \omega)) - P(s, \omega) u_2.$$

However without restriction on g it is not clear that constant controls are admissible. In our case we apply Assumption 2.8 (iv) as well as the quadratic growth of f and h together with Lemma 4.1 to see that indeed this is the case. □

6 The cost-adjusted target function

In Sect. 5 we have established a necessary and sufficient condition for the optimality of a given control. Unfortunately in the case of the passive order it is far from explicit. In the present section we prove Theorem 3.5, which provides a more explicit characterisation, and discuss further properties of the cost-adjusted target function as given in Definition 3.4.

Proposition 6.1 *There exist constants $c_{\tilde{\alpha}}$ and η such that for each $z \in \mathbb{R}^n$,*

$$\sup_{t \leq s \leq T} |\tilde{\alpha}(s, z)| \leq c_{\tilde{\alpha}}\left(1 + \|z\|^{\eta}_{\mathbb{R}^n}\right).$$

Proof Choosing the zero control $u \equiv 0$ and using the polynomial growth of the functions f, h and α as well as Lemma 4.1 we see that there exist c_1 and η_1 such that

$$v(t, 0, z) \leq J(t, 0, z, 0) \leq c_1 \left(1 + \|z\|_{\mathbb{R}^n}^{\eta_1}\right).$$

If we now apply Lemma 4.2 we find further constants $c_2 > 0$, c_3 and η_2 such that

$$v(t, \tilde{\alpha}(t, z), z) \geq c_2 |\tilde{\alpha}(t, z)|^2 - c_3 \left(1 + \|z\|_{\mathbb{R}^n}^{\eta_2}\right).$$

Since $\tilde{\alpha}$ is the pointwise minimiser of v with respect to x, combining the above inequalities and relabelling constants provides the result. □

Proposition 6.2 *The optimal passive order $\hat{u}_1$ is given $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$ by*

$$\hat{u}_1(s, \omega) = \tilde{\alpha}(s, Z(s-, \omega)) - \hat{X}(s-, \omega).$$

Proof We consider a process $\tilde{X}$ which solves the following SDE on $[t, T]$,

$$d\tilde{X}(s) = \hat{u}_2(s)ds + \left(\tilde{\alpha}(s, Z(s-)) - \tilde{X}(s-)\right) N(ds), \quad \tilde{X}(t) = x,$$

and want to show that the control $\tilde{u}$ defined for $s \in [t, T]$ by

$$\tilde{u}(s) \triangleq \begin{pmatrix} \tilde{\alpha}(s, Z(s-)) - \tilde{X}(s-) \\ \hat{u}_2(s) \end{pmatrix},$$

is admissible. The predictability and progressive measurability are straightforward and thus we need only check the L^2-nature of the control, which is a consequence of the following estimate,

$$\sup_{t \leq s \leq T} |\tilde{X}(s)|^2 \leq c \left(\int_t^T |\hat{u}_2(s)|^2 ds + \sup_{t \leq s \leq T} |\tilde{\alpha}(s, Z(s))|^2 \right),$$

together with Proposition 6.1 and Lemma 4.1.

Now let us prove that such a strategy is in fact optimal. We let τ_1 be the first jump time of N after t and $\tau \triangleq \tau_1 \wedge T$. By the dynamic programming principle we have

$$\begin{aligned} &\mathbb{E}_{t,x,z} \left[\int_t^\tau g(\hat{u}_2(s), Z(s)) + h(\tilde{X}(s) - \alpha(s, Z(s)))ds + v\left(\tau, \tilde{X}(\tau), Z(\tau)\right) \right] \\ &\geq \mathbb{E}_{t,x,z} \left[\int_t^\tau g(\hat{u}_2(s), Z(s)) + h(\hat{X}(s) - \alpha(s, Z(s)))ds + v\left(\tau, \hat{X}(\tau), Z(\tau)\right) \right]. \end{aligned}$$

On the stochastic time interval $[t, \tau)$ we have that the optimal trajectories $\hat{X}$ and $\tilde{X}$ coincide, and on the set $\{\tau_1 > T\}$ we have equality in the above. If we define the set $A \triangleq \{\tau_1 \leq T\}$ then the above inequality leads to

$$\begin{aligned} E_{t,x,z}\left[v\left(\tau_1, \tilde{X}(\tau_1), Z(\tau_1)\right) \mathbb{1}_A\right] &\geq \mathbb{E}_{t,x,z}\left[v\left(\tau_1, \hat{X}(\tau_1), Z(\tau_1)\right) \mathbb{1}_A\right] \\ &= \mathbb{E}_{t,x,z}\left[v\left(\tau_1, \hat{X}(\tau_1-) + \hat{u}_1(\tau_1), Z(\tau_1)\right) \mathbb{1}_A\right]. \end{aligned}$$

Independence of N and M together with [5, Proposition 1.3.12] implies $Z(\tau_1-) = Z(\tau_1)$ so that by the construction of the process $\tilde{X}$ we get

$$\mathbb{E}_{t,x,z}\left[v\left(\tau_1, \tilde{X}(\tau_1), Z(\tau_1)\right) \mathbb{1}_A\right] = \mathbb{E}_{t,x,z}\left[v\left(\tau_1, \tilde{\alpha}\left(\tau_1, Z(\tau_1)\right), Z(\tau_1)\right) \mathbb{1}_A\right].$$

Since $\tilde{\alpha}$ is the pointwise minimiser of the value function with respect to x, the statement of the proposition can now be derived by applying a contradiction argument together with the law of total expectation. □

For the subsequent analysis, including the proof of Theorem 3.5, we denote by

$$(\hat{X}^{t,x,z}, Z^{t,z}, P^{t,x,z})$$

the solution to the coupled FBSDE given by Eqs. 2.1, 2.2 and 3.1, started at $(t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$.

Lemma 6.3 *If $x < y$ then $\hat{X}^{t,x,z}(s) \leq \hat{X}^{t,y,z}(s)$ for each $s \in [t, T]$.*

Proof Denote by τ the first jump time of N after time t. If there is a jump in $[t, T]$ then by Proposition 6.2

$$\hat{X}^{t,x,z}(\tau) = \hat{X}^{t,y,z}(\tau) = \tilde{\alpha}(\tau, Z(\tau-)).$$

Due to the uniqueness of the solution to the FBSDE for any initial data we derive the flow property, exactly as in Pardoux and Tang [26, Theorem 5.1],

$$\hat{X}^{s,\hat{X}^{t,x,z}(s),Z^{t,z}(s)}(r) = \hat{X}^{t,x,z}(r), \quad t \leq s \leq r \leq T. \tag{6.1}$$

This implies that $\hat{X}^{t,x,z}$ and $\hat{X}^{t,y,z}$ coincide on $[\tau, T]$. Before a jump of N, $\hat{X}^{t,x,z}$ and $\hat{X}^{t,y,z}$ evolve continuously and we define the stopping time

$$\tau_2 \triangleq \inf \left\{ s \geq t : \hat{X}^{t,x,z}(s) = \hat{X}^{t,y,z}(s) \right\} \wedge \tau \wedge T.$$

By continuity $\hat{X}^{t,x,z} < \hat{X}^{t,y,z}$ in $[s, \tau_2)$ and $\hat{X}^{t,x,z} = \hat{X}^{t,y,z}$ in $[\tau_2, \tau \wedge T]$ again thanks to the flow property. □

The following result is classical in the study of fully coupled FBSDEs, see for instance Bender and Zhang [8, Corollary 6.2]. Since we have jumps we provide a proof.

Proposition 6.4 *There exists a deterministic measurable function $\varphi : [t, T] \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ such that for $s \in [t, T]$ we have*

$$P^{t,x,z}(s) = \varphi(s, \hat{X}^{t,x,z}(s), Z^{t,z}(s)).$$

Proof When $t = 0$ since $P^{0,x,z}$ is adapted and the filtration is generated by the (compound) Poisson processes and the Brownian motion we have that $P^{0,x,z}(0)$ is constant so that the map $(x, z) \mapsto P^{0,x,z}(0)$ is well defined. Using a time shift argument exactly as in El Karoui et al. [16, Proposition 4.2] one can show that $P^{t,x,z}(t)$ is deterministic so that the map

$$\varphi(t, x, z) = P^{t,x,z}(t)$$

is well defined. Using the flow property (6.1) we see that for $s \in [t, T]$

$$P^{t,x,z}(s) = P^{s,\hat{X}^{t,x,z}(s),Z^{t,z}(s)}(s) = \varphi(s, \hat{X}^{t,x,z}(s), Z^{t,z}(s)),$$

as required. □

Proposition 6.5 *For all $s \in [t, T]$ and $z \in \mathbb{R}^n$ the map $x \mapsto \varphi(s, x, z)$ is strictly decreasing. Moreover we have $\varphi(s, \tilde{\alpha}(s, z), z) = 0$.*

Proof Using Proposition 6.4 and the definition of the adjoint equation (3.1) we have the representation

$$P^{t,x,z}(t) = \varphi(t,x,z) = -\mathbb{E}_{t,x,z}\left[\int_t^T h'\left(\hat{X}^{t,x,z}(s) - \alpha(s, Z^{t,z}(s))\right) ds\right] \\ -\mathbb{E}_{t,x,z}\left[f'\left(\hat{X}^{t,x,z}(T) - \alpha(T, Z^{t,z}(T))\right)\right]. \tag{6.2}$$

Suppose $x < y$, then from Lemma 6.3 together with the càdlàg property of the paths of $\hat{X}^{t,x,z}$ and the fact that h', f' are normalised and strictly increasing, it follows that $\varphi(s,x,z) \geq \varphi(s,y,z)$.

We observe that $\varphi(t,x,z) = \varphi(t,y,z)$ would imply $\hat{X}^{t,x,z}(s) = \hat{X}^{t,y,z}(s) ds \times d\mathbb{P}$ a.e. on $[t,T] \times \Omega$ so that $\hat{X}^{t,x,z}$ and $\hat{X}^{t,y,z}$ would be indistinguishable by Lemma A.1, which contradicts $\hat{X}^{t,x,z}(t) = x < y = \hat{X}^{t,y,z}(t)$.

To prove the second claim, let τ_1 denote the first jump time of N after t and define $\tau \triangleq \tau_1 \wedge T$. Due to the independence of N and M, we see from [5, Proposition 1.3.12] that they do not jump at the same time. In particular, using that τ_1 is exponentially distributed with parameter λ, we may write

$$\begin{aligned}\mathbb{E}\left[P(\tau)^2\right] &= \mathbb{E}\left[(P(\tau-) + R_1(\tau))^2\right] \\ &= \mathbb{E}\left[\int_t^T \lambda e^{-\lambda(s-t)} \left(P(s-) + R_1(s)\right)^2 ds\right] = 0.\end{aligned}$$

The final equality follows since $P(s-) + R_1(s) = 0$, $ds \times d\mathbb{P}$ a.e. on $[t,T] \times \Omega$ by Theorem 3.2 and we have dropped the superscripts as we now consider a fixed starting point $(t,x,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n$. Using Proposition 6.4 we may write this as

$$0 = \mathbb{E}\left[P(\tau)^2\right] = \mathbb{E}\left[\varphi\left(\tau, \hat{X}(\tau), Z(\tau)\right)^2\right] = \mathbb{E}\left[\int_t^T \lambda e^{-\lambda(s-t)} \varphi\left(s, \hat{X}(s), Z(s)\right)^2 ds\right].$$

A consequence of Proposition 6.2 is that

$$\hat{X}(\tau) = \hat{X}(\tau-) + \hat{u}_1(\tau) = \tilde{\alpha}(\tau, Z(\tau-)) = \tilde{\alpha}(\tau, Z(\tau)),$$

so we have

$$0 = \mathbb{E}\left[\int_t^T \lambda e^{-\lambda(s-t)} \varphi\left(s, \tilde{\alpha}(s, Z(s)), Z(s)\right)^2 ds\right],$$

and thus $\varphi\left(s, \tilde{\alpha}(s, Z(s)), Z(s)\right) = 0$ $ds \times d\mathbb{P}$ a.e. on $[t,T] \times \Omega$. Since the process P (and hence φ) is càdlàg, an argument as in Lemma A.1 now shows $\varphi\left(s, \tilde{\alpha}(s, Z(s)), Z(s)\right) = 0$ for all $s \in [t,T]$. □

We are now in a position to prove Theorem 3.5.

Proof of Theorem 3.5 Assertion (i) is the content of Proposition 6.2. To prove the second part, first recall the buy region

$$\mathcal{R}_{\text{buy}} \triangleq \left\{(s,x,z) \in [t,T] \times \mathbb{R} \times \mathbb{R}^n : x < \tilde{\alpha}(s,z)\right\}.$$

Using Proposition 6.5 we conclude that for (s, ω) such that $(s, \hat{X}(s-, \omega), Z(s-, \omega))$ is in the buy region we have $P(s, \omega) > 0$, recall that $P(s-, \omega)$ and $P(s, \omega)$ are equal $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$. From Theorem 3.2 $\hat{u}_2$ is the pointwise minimiser of $u_2 \mapsto g(u_2, z) - Pu_2$, $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$. Inspection shows that the minimiser of this function is positive, which gives the required property of $\hat{u}_2$. The fact that $\hat{u}_1 > 0$ $ds \times d\mathbb{P}$ a.e. for such (t, ω) is a consequence of the definition of the buy region together with Proposition 6.2 . The proof for the sell region is symmetric. □

Constructing the cost-adjusted target function is key to solving the curve following problem as it characterises the buy and sell regions and we now provide further analysis. We show that the map $\alpha \mapsto \tilde{\alpha}$ is translation invariant and preserves orderings and boundedness. Moreover, in the case when α is a deterministic function $\tilde{\alpha}$ coincides with α if and only if α is constant. We first demonstrate that the map $\alpha \mapsto \tilde{\alpha}$ is monotone. This property is natural, a larger target cannot correspond to a smaller cost-adjusted target.

Proposition 6.6 *If $\alpha(s, z) \geq \beta(s, z)$ for all $(s, z) \in [t, T] \times \mathbb{R}^n$ then we have*

$$\tilde{\alpha}(s, z) \geq \tilde{\beta}(s, z) \quad \textit{for all } (s, z) \in [t, T] \times \mathbb{R}^n.$$

Proof We prove the claim by contradiction and assume there exists $(t_0, z_0) \in [t, T] \times \mathbb{R}^n$ with $\tilde{\alpha}(t_0, z_0) < \tilde{\beta}(t_0, z_0)$ so that one may choose x_0 with

$$\tilde{\alpha}(t_0, z_0) < x_0 < \tilde{\beta}(t_0, z_0).$$

We denote by $(\hat{X}^\alpha, P^\alpha)$ and $(\hat{X}^\beta, P^\beta)$ the optimal pairs for the problem started at (t_0, x_0, z_0) with targets α and β, respectively.

Observe first that by Proposition 6.5 we have $P^\alpha(t_0) < 0 < P^\beta(t_0)$. Define the stopping time

$$\tau_{\alpha,\beta} \triangleq \inf\left\{s \in [t_0, T] \ : \ P^\alpha(s) \geq P^\beta(s)\right\}$$

as the first time that P^α is larger than or equal to P^β, with the convention $\inf \emptyset = \infty$. To deduce a contradiction we must first establish several properties of the stopping time $\tau_{\alpha,\beta}$.

If τ_1 denotes, as in Lemma 4.3, the first jump time after t_0 of the Poisson process N then from Propositions 6.2 and 6.5 we have $P^\alpha(\tau_1) = P^\beta(\tau_1) = 0$. Thus we conclude

$$\tau_{\alpha,\beta} \leq \tau_1. \tag{6.3}$$

The waiting time until the first jump of N or M is exponentially distributed and since P^α, P^β evolve continuously in the absence of jumps we have

$$\tau_{\alpha,\beta} > t_0. \tag{6.4}$$

We now want to compare the processes $\hat{X}^\alpha$ and $\hat{X}^\beta$ up to the stopping time $\tau_{\alpha,\beta}$. The function $g(\cdot, z)$ is assumed to be smooth and strictly convex for fixed $z \in \mathbb{R}^n$, in addition it has uniform quadratic growth in u_2. This implies that $g_{u_2}(\cdot, z)$ is invertible with a well defined strictly increasing inverse, for all $z \in \mathbb{R}^n$. Thus we deduce that $ds \times d\mathbb{P}$ a.e. on $(t_0, \tau_{\alpha,\beta} \wedge T) \times \Omega$

$$\hat{u}_2^\alpha(s) = [g_{u_2}(\cdot, Z(s))]^{-1}(P^\alpha(s)) \leq [g_{u_2}(\cdot, Z(s))]^{-1}(P^\beta(s)) = \hat{u}_2^\beta(s).$$

Using (6.3) we have that for $s \in [t_0, \tau_{\alpha,\beta})$

$$\hat{X}^\alpha(s) = x_0 + \int_{t_0}^{s} \hat{u}_2^\alpha(r)dr \leq x_0 + \int_{t_0}^{s} \hat{u}_2^\beta(r)dr = \hat{X}^\beta(s), \tag{6.5}$$

which shows that

$$\hat{X}^{\alpha}(s) - \alpha(s, Z(s)) \leq \hat{X}^{\beta}(s) - \beta(s, Z(s)). \tag{6.6}$$

Finally, consider the set $\{\tau_{\alpha,\beta} > T\}$. On this set we have $P^{\alpha}(T) < P^{\beta}(T)$, thus using the terminal condition of the BSDE (3.1) we see

$$f'\left(\hat{X}^{\alpha}(T) - \alpha(T, Z(T))\right) > f'\left(\hat{X}^{\beta}(T) - \beta(T, Z(T))\right).$$

However comparing this with (6.6) and noting $\tau_{\alpha,\beta} > T$ we get a contradiction as f' is strictly increasing. Thus we also have

$$\tau_{\alpha,\beta} \leq T. \tag{6.7}$$

Let us now derive a contradiction. We write using (6.6)

$$\begin{aligned}
\mathbb{E}_{t_0,x_0,z_0}\left[P^{\alpha}(\tau_{\alpha,\beta}) - P^{\alpha}(t_0)\right] &= \mathbb{E}_{t_0,x_0,z_0}\left[\int_{t_0}^{\tau_{\alpha,\beta}} h'\left(\hat{X}^{\alpha}(s) - \alpha(s, Z(s))\right) ds\right] \\
&\leq \mathbb{E}_{t_0,x_0,z_0}\left[\int_{t_0}^{\tau_{\alpha,\beta}} h'\left(\hat{X}^{\beta}(s) - \beta(s, Z(s))\right) ds\right] \\
&= \mathbb{E}_{t_0,x_0,z_0}\left[P^{\beta}(\tau_{\alpha,\beta}) - P^{\beta}(t_0)\right].
\end{aligned}$$

This implies

$$\mathbb{E}_{t_0,x_0,z_0}\left[P^{\alpha}(\tau_{\alpha,\beta}) - P^{\beta}(\tau_{\alpha,\beta})\right] \leq P^{\alpha}(t_0) - P^{\beta}(t_0) < 0,$$

but $P^{\alpha}(\tau_{\alpha,\beta}) \geq P^{\beta}(\tau_{\alpha,\beta})$ by definition, which is the desired contradiction. □

Next, we show that the map $\alpha \mapsto \tilde{\alpha}$ is translation invariant.

Proposition 6.7 *For any constant c, if $\beta = \alpha + c$ then $\tilde{\beta} = \tilde{\alpha} + c$.*

Proof Let us denote by v^{α} and v^{β} the value functions corresponding to the targets α and β, respectively. We use that $u \mapsto X^u$ is affine to deduce

$$\begin{aligned}
v^{\alpha}(t, x, z) = \inf_{u \in \mathcal{U}_t} \mathbb{E}_{t,x,z}\Bigg[\int_t^T & g(u_2, Z(s)) + h\left(X^u(s) + c - \alpha(s, Z(s)) - c\right) ds \\
& + f\left(X^u(T) + c - \alpha(T, Z(T)) - c\right)\Bigg] \\
= v^{\beta}(t, x + c, z).&
\end{aligned}$$

where the final line follows from using translation properties of the expectation together with the definition of v^{β}. Since the cost-adjusted target is defined to be

$$\tilde{\alpha}(t, z) = \arg\min_{x \in \mathbb{R}} v^{\alpha}(t, x, z),$$

the result follows. □

In the case that α is independent of z, we can say more about the structure of the cost-adjusted target. This is the content of the following proposition.

Proposition 6.8 *Let α be independent of z and continuously differentiable in t. Then $\tilde{\alpha} \equiv \alpha$ if and only if α is constant.*

Proof Suppose $\alpha \equiv c$, a constant. If $x = c$ then the control $u \equiv 0$ yields $J(t, x, z, u) = 0$, where we use the notation of Sect. 2. Since J is nonnegative we see that $v(t, c, z) = 0$ and that $v(t, x, z) > 0$ for $x \neq c$. This implies that $\tilde{\alpha}(t, z) = \arg\min_x v(t, x, z) = c$.

We prove the opposite implication by contradiction. Suppose that α is not constant, by continuity there is a global minimum and maximum. At least one of them is attained at $t_0 > 0$, and we only consider the case that α attains a maximum at t_0 (the case of a minimum is symmetric). Now there is $\delta > 0$ such that α is strictly increasing on $[t_0 - \delta, t_0]$. For the remainder of the proof we assume $s \in [t_0 - \delta, t_0)$.

We denote by $\hat{X}$ the process $\hat{X}^{t_0-\delta,\alpha(t_0-\delta),z}$ and P the corresponding solution to the backward equation. The crucial observation is that

$$\hat{X}(s) \leq \alpha(s), \tag{6.8}$$

for each $s \in [t_0 - \delta, t_0)$. Indeed, if τ denotes a jump time of the Poisson process N, then by Proposition 6.2 and the assumption $\tilde{\alpha} \equiv \alpha$ we have

$$\hat{X}(\tau) = \tilde{\alpha}(\tau, Z(\tau-)) = \alpha(\tau).$$

In particular $\hat{X}$ does not jump above α on $[t_0 - \delta, t_0)$.

Furthermore, if there is no jump and we have $\hat{X}(s) = \alpha(s) = \tilde{\alpha}(s)$ then by Theorem 3.5(iii) we have $ds \times d\mathbb{P}$ a.e. on $[t_0 - \delta, t_0] \times \Omega$

$$\hat{u}_2(s, \omega) = 0 < \alpha'(s),$$

as α is smooth and strictly increasing on $[t_0 - \delta, t_0)$ by assumption. The implication is that $\hat{X}$ does not cross α from below and (6.8) holds.

As $\hat{X}$ is never above $\tilde{\alpha}$ Proposition 6.5 implies

$$P(s) \geq 0, \tag{6.9}$$

for $s \in [t_0 - \delta, t_0)$. However from the definition of P we have

$$\mathbb{E}_{t_0-\delta,\alpha(t_0-\delta),z}\left[P(s) - P(t_0 - \delta)\right] = \mathbb{E}_{t_0-\delta,\alpha(t_0-\delta),z}\left[\int_{t_0-\delta}^{s} h'\left(\hat{X}(r) - \alpha(r)\right) dr\right] \leq 0.$$

The last inequality follows from noting $\hat{X}(r) - \alpha(r) \leq 0$ and $h'(0) = 0$, which is itself a consequence of Assumption 2.6 and the normalisation of h.

Rearranging the above inequality we see that

$$\mathbb{E}_{t_0-\delta,\alpha(t_0-\delta),z}\left[P(s)\right] \leq P(t_0 - \delta). \tag{6.10}$$

In addition we have $P(t_0 - \delta) = 0$, since $\hat{X}$ starts on the cost-adjusted target function. Combining (6.9) and (6.10) we now see that $P(s) = 0$ for $s \in [t_0 - \delta, t_0)$.

An application of Theorem 3.2 now shows that we have $\hat{u}_2(s) = 0$ $ds \times d\mathbb{P}$ a.e. on $[t_0 - \delta, t_0)$, thus $\hat{X}$ has paths which are almost surely constant on this interval.

However since $P(s) = 0$, Proposition 6.5 implies $\hat{X}(s) = \tilde{\alpha}(s) = \alpha(s)$ for each $s \in [t_0 - \delta, t_0)$, which is by assumption strictly increasing, this provides the necessary contradiction. □

As a corollary we show that, when acting on bounded functions, the map $\alpha \mapsto \tilde{\alpha}$ gives again bounded functions. This is natural, the cost-adjusted target should not exceed the maximum of the target function.

Corollary 6.9 *Let $(s, z) \in [t, T] \times \mathbb{R}^n$. We have the following estimate,*

$$\inf_{t \le r \le T} \inf_{y \in \mathbb{R}^n} \alpha(r, y) \le \tilde{\alpha}(s, z) \le \sup_{t \le r \le T} \sup_{y \in \mathbb{R}^n} \alpha(r, y).$$

Proof We only prove the first inequality and define

$$c \triangleq \inf_{t \le r \le T} \inf_{y \in \mathbb{R}^n} \alpha(r, y).$$

If $c = -\infty$, there is nothing to prove. Since the function α has polynomial growth, we may assume $c \in \mathbb{R}$. From Proposition 6.6 we have $\tilde{\alpha}(s, z) \ge \tilde{c}$ for all $(s, z) \in [t, T] \times \mathbb{R}^n$ and $\tilde{c} = c$ from Proposition 6.8. □

We now move on to consider some special cases.

7 Examples

In this section, we shall show that Theorem 3.2 may be used to derive a closed form solution for the optimal control when the penalty and cost functions are quadratic. In this case our problem becomes one of quadratic linear regulator type, which have been well studied in the literature, see Yong and Zhou [34, Chap. 6] for an overview. The novelty in the present applications is the interpretation of the jumps in terms of passive order execution.

7.1 Curve following with signal

The following proposition gives an example under which we can find the optimal control (semi-)explicitly.

Proposition 7.1 *Let $t = 0$ and $g(u_2, z) = \kappa u_2^2$ for a constant $\kappa > 0$, $h(x) = f(x) = x^2$ and α be any continuous function satisfying Assumption 2.6. Suppose that the signal is defined by*

$$dZ(s) = \mu(s, Z(s))ds + \sigma(s, Z(s))dW(s), \quad Z(0) = z.$$

where μ and σ are bounded continuous real-valued functions with $\sigma \ge \delta > 0$ for some constant δ. Suppose in addition that μ, σ, α are Hölder continuous for some exponent less than 1, uniformly in t. Then the optimal control is given $ds \times d\mathbb{P}$ a.e by

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-), Z(s)) = -\dfrac{b(s, Z(s))}{a(s)} - \hat{X}(s-), \\ \hat{u}_2(s, \hat{X}(s), Z(s)) \quad = -\dfrac{a(s)}{2\kappa}\left(-\dfrac{b(s, Z(s))}{a(s)} - \hat{X}(s)\right), \end{cases} \tag{7.1}$$

where the functions a and b satisfy linear PDEs and can be given in (semi)explicit form.

Proof We first note that one can check by direct computation that the functions $g(u_2, \cdot) = \kappa u_2^2$ and $f(x) = h(x) = x^2$ satisfy Assumptions 2.6 and 2.8. We know from Theorem 3.2 that $ds \times d\mathbb{P}$ a.e. on $[0, T] \times \Omega$, the optimal market order is given by the pointwise minimiser of

$$u_2 \mapsto \kappa u_2^2 - P(s)u_2.$$

Thus we compute that $u_2(s) = (2\kappa)^{-1} P(s)$. This leads to the following coupled FBSDE,

$$\begin{cases} d\hat{X}(s) = (2\kappa)^{-1} P(s)ds + \hat{u}_1(s)N(ds), \\ dP(s) = 2\left(\hat{X}(s) - \alpha\,(s, Z(s))\right) ds + Q(s)dW(s) + R_1(s)\tilde{N}(ds), \\ \hat{X}(0) \;= x,\; P(T) = -2\left(\hat{X}(T) - \alpha\,(T, Z(T))\right). \end{cases} \tag{7.2}$$

We have omitted the process Z in the above as it may be solved independently of $(P, \hat{X})$ and then regarded as an input.

It follows from Theorem 3.2 that any solution of the above for which $P(s-) + R_1(s) = 0$, $ds \times d\mathbb{P}$ a.e. on $[0, T] \times \Omega$ provides the optimal control. Moreover, since the coupled FBSDE is in one to one correspondence with the optimal control (again by Theorem 3.2) there is at most one solution. We make the ansatz

$$P(s) = a\,(s, Z(s))\,\hat{X}(s) + b\,(s, Z(s)), \tag{7.3}$$

for deterministic functions a and b to be determined.

A consequence of the above ansatz is that the jumps of P are equal to a times the jumps of $\hat{X}$. In particular we know that $\hat{u}_1$ should ensure

$$P(s-) + R_1(s) = a\,(s, Z(s))\,[\hat{X}(s-) + \hat{u}_1(s)] + b\,(s, Z(s)) = 0.$$

This gives the following form for the passive order,

$$\hat{u}_1\left(s, \hat{X}(s-), Z(s)\right) = -\frac{b(s, Z(s))}{a\,(s, Z(s))} - \hat{X}(s-).$$

Inserting this into (7.2) leads to the FBSDE which we hope to solve by the above ansatz. We proceed similarly to the four step scheme of Ma, Protter and Yong [23], applying Itô's formula to the expression for P from Eq. 7.3 and using the definition of the optimal controls we derive that

$$\begin{aligned} dP = {} & \hat{X}\left(a_s + \mu a_z + \frac{1}{2}\sigma^2 a_{zz}\right) ds + \sigma\left(\hat{X} a_z + b_z\right) dW(s) - \left(\frac{b}{a} + \hat{X}\right) d\tilde{N}(s) \\ & + \left(\frac{1}{2\kappa} a\left[a\hat{X} + b\right] - \lambda b - \lambda a \hat{X}\right) ds \\ & + \left(b_s + \mu b_z + \frac{1}{2}\sigma^2 b_{zz}\right) ds, \end{aligned}$$

where we suppress the arguments (s, z) of a, b, μ, σ and $\hat{X}$ for brevity. These dynamics must coincide with those of Eq. 7.2 so that matching the coefficients we derive that the functions a and b should solve the following partial differential equations on $[0, T] \times \mathbb{R}$,

$$a_s + \mu a_z + \frac{1}{2}\sigma^2 a_{zz} - \lambda a + \frac{1}{2\kappa} a^2 - 2 = 0, \quad a(T, z) = -2, \tag{7.4}$$

$$b_s + \mu b_z + \frac{1}{2}\sigma^2 b_{zz} - \lambda b + \frac{1}{2\kappa} ab + 2\alpha = 0, \quad b(T, z) = 2\alpha(T, z), \tag{7.5}$$

where again we suppress the (s, z) for notational simplicity. Equation 7.4 for a can be solved independently of z as a standard Riccati equation,

$$a(s) = \kappa\left(\lambda + \zeta - \frac{2\zeta}{1 - c_a e^{\zeta s}}\right), \tag{7.6}$$

where we define $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa}$ and choose c_a such that the boundary condition at T is satisfied. A calculation shows that $a(s) < 0$ for all $s \in [0, T]$, so the terms in (7.1) are well defined. The PDE (7.5) for b then becomes a heat equation with a source term and bounded Hölder continuous coefficients for which Friedman [18, Theorem 1.7.12] gives the existence of a solution as well as a semi-explicit formula in terms of Green's functions. Using the Feynman–Kac formula we can give a more explicit probabilistic solution which better displays its relation with the original problem. Namely we have

$$b(s,z) = 2\mathbb{E}_{s,z}\left[\int_s^T \exp\left(\int_s^r \rho(w)dw\right)\alpha(r, Z(r))dr + \exp\left(\int_s^T \rho(w)dw\right)\alpha(T, Z(T))\right],$$

where we use the notation $\mathbb{E}_{s,z}[\cdot]$ as in Lemma 4.2 and the function $\rho(s) \triangleq \frac{a(s)}{2\kappa} - \lambda$ for $s \in [0, T]$. □

Remark 7.2 In the case of quadratic cost functions there is a clear economic interpretation of the controls. The function b encodes the expected future motion, appropriately discounted by λ and $\frac{a}{2\kappa}$, of the target function with respect to the distribution of the signal Z. This then feeds into the cost-adjusted target function via the ratio $\frac{b}{a}$. We see that this is forward looking, evolves in time and reacts to changes in Z. The trading is described thus, a passive order is placed and continuously adjusted so that when a jump occurs the stock holdings move to the cost-adjusted target function. Simultaneously market orders are used with a rate $\frac{-a(s)}{2\kappa}$ proportional to the amount held in the passive order.

Such a control structure is intuitive, for fixed (s, z) trading slows down as one approaches the cost-adjusted target $\tilde{\alpha}$ and speeds up as one moves away. Put another way, when the agent has stock holdings near the function $\tilde{\alpha}$ he reduces the trading in market orders, preferring to wait for passive order execution. The parameter κ describes how expensive trading is, when it is large the agent uses less market orders and relies more on passive orders. This again coincides with trading strategies seen in markets with low liquidity.

We point out a final point on the nature of the solution. We have chosen the function $f(x) = \eta x^2$, with the value $\eta = 1$. For general η one can simply scale the value function by η^{-1} to reduce to the current setting, with parameter $\kappa' = \frac{\kappa}{\eta}$. Now we can interpret η as urgency parameter, a key feature present in all modern trading algorithms, this controls how close one should adhere to the target and thus decides the allocation between market and passive orders.

7.2 Portfolio liquidation

We consider an investor who wants to sell x stock shares over the interval $[0, T]$, however trading incurs liquidity costs. The question then becomes what the optimal strategy should be. The construction of such a trading program has received much attention recently, see for instance Almgren and Chriss [3], Obizhaeva and Wang [24] as well as Schied and Schöneborn [31]. With a little extra work one can embed this in the present setup. When we assume, as in [1], that market orders incur quadratic trading costs and there is a quadratic penalty on stock holdings we are led to the following formulation,

$$v(0,x) \triangleq \inf_{u\in\mathcal{U}_0, X^u(T)=0} \mathbb{E}_{0,x}\left[\int_0^T \kappa|u_2(s)|^2 + |X^u(s)|^2 ds\right],$$

where $\kappa > 0$ and we have omitted a signal for ease of exposition. Observe the new feature in the present optimisation problem is that we now have a binding constraint, we are interested in only those controls for which $X^u(T) = 0$.

Proposition 7.3 *The optimal control is given $ds \times d\mathbb{P}$ a.e. by*

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-)) = -\hat{X}(s-), \\ \hat{u}_2(s, \hat{X}(s)) \quad = \dfrac{a(s)}{2\kappa}\hat{X}(s), \end{cases}$$

where a is given by

$$a(s) = \kappa\left(\lambda + \zeta - \frac{2\zeta}{1 - e^{\zeta(s-T)}}\right), \tag{7.7}$$

with $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa}$.

Proof Let $n \in \mathbb{N}$ and define the sequence of approximate value functions

$$v^n(0,x) \triangleq \inf_{u\in\mathcal{U}_0} \mathbb{E}_{0,x}\left[\int_0^T \kappa|u_2(s)|^2 + |X^u(s)|^2 ds + n|X^u(T)|^2\right].$$

Applying the methods of the previous subsection we see that the optimal control corresponding to v^n is given $ds \times d\mathbb{P}$ a.e. by

$$\begin{cases} \hat{u}_1^n(s, \hat{X}^n(s-)) = -\hat{X}^n(s-), \\ \hat{u}_2^n(s, \hat{X}^n(s)) \quad = \dfrac{a^n(s)}{2\kappa}\hat{X}^n(s), \end{cases}$$

where the a^n are defined by

$$a^n(s) \triangleq \kappa\left(\lambda + \zeta - \frac{2\zeta}{1 - c_n e^{\zeta(s-T)}}\right),$$

with the constant c_n chosen so that $a^n(T) = -2n$. A calculation shows that the functions a^n converge towards a and hence the optimal controls $\hat{u}^n$ converge to $\hat{u}$, for a and $\hat{u}$ given in the statement of the proposition. We may rewrite the portfolio liquidation problem as

$$v(0,x) = \inf_{u\in\mathcal{U}_0} \mathbb{E}_{0,x}\left[\int_0^T \kappa|u_2(s)|^2 + |X^u(s)|^2 ds + \delta_{\{\mathbb{R}\setminus\{0\}\}}\left(X^u(T)\right)\right],$$

where $\delta_{\{\mathbb{R}\setminus\{0\}\}}$ is the indicator function in the sense of convex analysis. This leads to the following inequality,

$$v^n(0,x) \le v(0,x) \quad \text{for all } x \in \mathbb{R}.$$

The optimality of the control given in the proposition will follow once we show that we have $\hat{X}(T) = 0$ for $\hat{X}$ satisfying

$$d\hat{X}(s) = \frac{a(s)}{2\kappa}\hat{X}(s)ds - \hat{X}(s-)N(ds), \quad \hat{X}(0) = x.$$

If N jumps in $[0, T]$ we clearly end up with zero stock holdings, thus we are reduced to showing that

$$\hat{X}(T) = x \exp\left(\frac{1}{2\kappa}\int_0^T a(s)ds\right)\mathbb{1}_{T\le\tau_1} = 0,$$

where τ_1 is again the first jump time of N. Using the explicit formula for a from (7.7), this equality can be verified and the proof of the proposition is finished. □

Remark 7.4 Let us analyse the optimal controls and stock holdings trajectory for this specific example. Since $s \in [0, T]$ we have the following approximation,

$$a(r) \approx \kappa\left(\lambda + \zeta - 2\zeta\left(1 + e^{\zeta(r-T)}\right)\right).$$

This leads to

$$\frac{1}{2\kappa}\int_0^s a(r)dr \approx \frac{1}{2\kappa}\left(\kappa(\lambda - \zeta)s + 2e^{-\zeta T}\left[e^{\zeta s} - 1\right]\right).$$

Observe that the term $e^{\zeta(s-T)}$ will be approximately constant for reasonable values of κ and λ. In particular when there are no jumps this leads to a first order approximation for the optimal stock holdings of

$$X(s) \approx x c_x e^{\frac{1}{2}(\lambda-\zeta)s}$$

for a constant c_x. It follows from the definition of ζ that $\lambda - \zeta < 0$, thus the optimal trajectory resembles the decreasing function e^{-s}. This corresponds with economic intuition, we expect to trade out of the stock holdings. Let us now motivate the convex shape. The cost terms are initially dominated by the deviation from zero and the agent sells at rate $\lambda - \zeta$ multiplied by the level of stock holdings. This results in the number of shares decreasing faster at first. Then, as holdings decrease, we see that the rate of decrease slows down. Observe further that the magnitude of $\lambda - \zeta$ is inversely proportional to κ so that the rate at which we sell is additionally influenced by the cost of market orders. If $\kappa = \frac{\kappa'}{\eta}$ as in Remark 7.2, then the key constant becomes the trade-off between cost and urgency. We note finally that the results here appear in the discrete time setting of P. Kratz and T. Schöneborn (unpublished, Proposition 4.2).

Remark 7.5 Proposition 7.3 solves the portfolio liquidation problem under strong assumptions (constant parameters, no spread, quadratic costs, quadratic penalty). Our main results, Theorems 3.2 and 3.5, can be used to obtain a solution under much weaker conditions, e.g. for general cost and penalty functions and a general stochastic signal Z. The process Z could represent e.g. the order book height, thereby extending [1] to a setting with stochastic liquidity parameters as well as passive orders.

8 Bid-ask spread and the independence of the jump processes

One choice for the stochastic signal Z would be bid ask spread. However, in our model a jump of N represents a liquidity event which executes the investor's passive order. In real markets a liquidity event which executes passive orders might also temporarily widen the bid ask spread on one side of the book. This is in contrast with our requirement that Z and N be independent. When one relaxes this assumption an interesting feature occurs (trading on different sides of the market) and we discuss this further in the present section.

Suppose that the dynamics of Z introduced in (2.2) are replaced by

$$dZ(s) = \mu(s, Z(s))ds + \sigma(s, Z(s))dW(s) + \int_{\mathbb{R}^k} \gamma(s, Z(s-), \theta)\, \tilde{M}(ds, d\theta) + \delta(s, Z(s-))\, N(ds),$$

for some function $\delta : [t, T] \times \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz in z, uniformly in s. Using an identical proof to that of Theorem 3.1 one can show existence and uniqueness of a solution as well as the characterisation of Theorem 3.2(ii). In this section we present examples to show that Theorem 3.5(ii) is no longer valid, in particular it can be optimal to place both market buy (sell) and passive sell (buy) orders simultaneously.

When one interprets the process $\hat{u}_1$ as an order placed in a crossing network, the above behaviour amounts to selling in the dark venue and buying in the visible venue (or vice versa). This phenomenon arises because when the passive order is executed, the signal jumps as well and the passive order "foresees" this, whereas the market order does not so that they may have different signs.

However when $\hat{u}_1$ is interpreted as a limit order this behaviour is equivalent to the investor placing liquidity on the buy (sell) side and then consuming it themselves. This is rather counterintuitive and is not economically rational or realistic, thus we conclude from our examples that in the general setting it is necessary to retain the assumption of independence between N and Z. Hence for arbitrary target functions Z may not be interpretable as spread. However by imposing specific conditions on α we show that, even when independence does not hold, the optimal control does not exhibit the undesirable behaviour described above. Thus in certain circumstances an interpretation of Z as spread is compatible with the notion of $\hat{u}_1$ as a limit order.

Our first example shows that if the signal enters the target but not the cost function then one may simultaneously buy and sell with positive probability.

Example 8.1 Consider the optimisation problem

$$v(t, x, z) \triangleq \inf_{u \in \mathcal{U}_t} \mathbb{E}_{t,x,z} \left[\int_t^T \kappa |u_2(s)|^2 + \left(X^u(s) - Z(s)\right)^2 ds \right],$$

where the target function is given by $\alpha(t, z) = z$, the stochastic signal satisfies

$$dZ(s) = -Z(s-)N(ds), \quad Z(t) = z,$$

the parameter $\kappa > 0$ and X^u has dynamics given by (2.1).

Applying the same ideas as in the proof of Proposition 7.1 it can be verified that the optimal control is given $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$ by

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-), Z(s-)) = -\dfrac{b(s, 0)}{a(s)} - \hat{X}(s-), \\ \hat{u}_2(s, \hat{X}(s-), Z(s-)) = -\dfrac{a(s)}{2\kappa}\left(Z(s-) - \hat{X}(s-)\right), \end{cases}$$

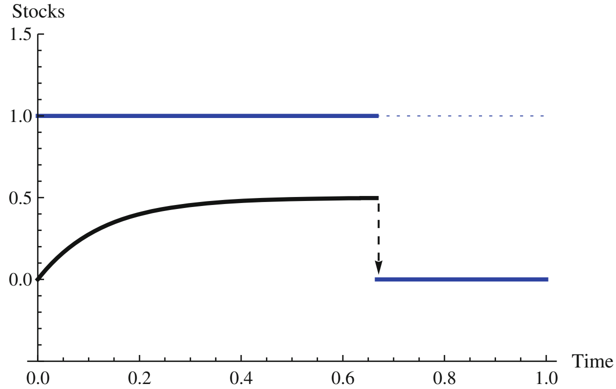

Fig. 1 The target function starts at $Y = 1$. Before the jump time market buy orders are used to reduce deviation, this is indicated by the line which starts at (0, 0) and is initially increasing. Note that before the jump time a passive sell order is placed such that stock holdings and signal both move to $Y = 0$ when the jump occurs

where the coefficients $a, b : [t, T] \mapsto \mathbb{R}$ are given by the following differential equations

$$\begin{cases} a_s - \lambda a + \frac{1}{2\kappa} a^2 - 2 \quad = 0, \quad a(T) = 0, \\ b_s - \lambda b + \frac{1}{2\kappa} ab + 2z = 0, \quad b(T, z) = 0. \end{cases}$$

The explicit solutions are given by

$$a(s) = \kappa \left(\lambda + \zeta - \frac{2\zeta}{1 - c_a e^{\zeta s}} \right),$$

$$b(s, z) = -a(s)z = -\kappa z \left(\lambda + \zeta - \frac{2\zeta}{1 - c_a e^{\zeta s}} \right),$$

where the integration constant c_a is chosen such that the terminal condition $a(T) = 0$ is met and we set $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa}$. Using the known form of a and b allows us to represent the optimal control before the first jump of N explicitly as

$$\hat{u}_1(s) = -\hat{X}(s-) \text{ and } \hat{u}_2(s) = -\frac{a(s)}{2\kappa} \left(z - \hat{X}(s-) \right).$$

The economic interpretation of the optimal controls is identical with that of Remarks 7.2 and 7.4 save for where the cost-adjusted target function is evaluated after the jump time of N. In particular, the new feature for the passive order is that in contrast with (7.1) it depends upon the signal Z evaluated not *before*, but *after* the jump of N, i.e. the investor places a passive order so that stock holdings jump to zero at a jump of N. Suppose now that $0 < \hat{X}(0) = x < z$, by inspecting the form of the optimal control one can see that before a jump of N we have $0 < \hat{X}(s-) < z$. As a consequence, $\hat{u}_1 < 0$ and $\hat{u}_2 > 0$, i.e. both market buy and passive sell orders are used and thus the orders do not have the same sign. A typical optimal trajectory is illustrated in Fig. 1 for the problem started at $t = 0$.

One might conjecture that such counterintuitive behaviour can be excluded in a model where the signal affects only the liquidity costs and further conditions are placed upon the target function, e.g. monotonicity. The following example shows this is not the case; in addition we provide an explicit solution to a curve following problem with a regime shift in liquidity.

Example 8.2 Consider the optimisation problem

$$v(t,x,z) = \inf_{u\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\left[\int_t^T Z(s)|u_2(s)|^2 + \big(\alpha(s) - X^u(s)\big)^2\, ds\right],$$

where X^u is as before, α is a deterministic function satisfying Assumption 2.6 and Z has dynamics

$$dZ(s) = (\kappa_2 - Z(s-))\,N(ds), \quad Z(t) = z = \kappa_1.$$

In this example the process Z has a nice economic interpretation as the inverse order book height. For $\kappa_1 > \kappa_2$ the jump of Z indicates a move into a regime where trading is cheap, or more expensive when $\kappa_2 > \kappa_1$. This model is a first order approximation to the effect on the curve following problem when there is the possibility of large orders being placed or cancelled on the best bid and ask simultaneously during the time horizon. Note that after the liquidity event, the jump of the Poisson process, the process Z moves to $\kappa_2 > 0$ and remains there until T. Observe that since Z is uniformly bounded from below Assumption 2.6 (ii) is satisfied.

First we show how to construct the solution to this problem, the subsequent analysis then demonstrates that the optimal control simultaneously buys and sells for an appropriate choice of monotonically decreasing time dependent target.

We are interested in $v(0,x,z)$, since Z takes only two values κ_1 and κ_2 we may view this as a value function in (t,x), with (κ_1,κ_2) as parameters as we now describe. Via the dynamic programming principle we may write

$$v(0,x) = \inf_{u\in\mathcal{U}_{0\tau}} \mathbb{E}_{0,x}\left[\int_0^\tau \kappa_1|u_2(s)|^2 + \big(\alpha(s) - X^u(s)\big)^2\, ds + \overline{v}(\tau, X^u(\tau-) + u_1(\tau))\right],$$

where τ is equal to $\tau_1 \wedge T$, τ_1 being the first jump time of N, $\mathcal{U}_{0\tau}$ is the control set restricted to $[0,\tau]$ and $\overline{v}$ is the value function

$$\overline{v}(t,x) \triangleq \inf_{u\in\mathcal{U}_t} \mathbb{E}_{t,x}\left[\int_t^T \kappa_2|u_2(s)|^2 + \big(\alpha(s) - X^u(s)\big)^2\, ds\right].$$

Exactly as in Pham [27, Sect. 3.6.2] we may use the explicit formula for the distribution of τ together with the law of total expectation and a change in the order of integration to write

$$v(0,x) = \inf_{u\in\mathcal{U}_0} \mathbb{E}_{0,x} \times\left[\int_0^T \beta(s)\Big[\kappa_1|u_2(s)|^2 + \big(\alpha(s) - X^u(s)\big)^2 + \lambda\overline{v}(s, X^u(s-) + u_1(s))\Big]\, ds\right],$$

where we used the definition $\beta(s) \triangleq e^{-\lambda s}$ to ease the notation.

The function $\overline{v}$ can be seen to be the value function of a linear quadratic regulator problem and as such can be shown via a standard verification theorem to be given by

$$\overline{v}(t,x) = \frac{1}{2}\overline{a}(t)x^2 + \overline{b}(t)x + \overline{c}(t).$$

The functions $\overline{a}, \overline{b}, \overline{c} : [0, T] \mapsto \mathbb{R}$ solve the following system of Riccati equations on $[0, T]$

$$\begin{cases} \bar{a}_s - \frac{1}{2\kappa_2}\bar{a}^2 + 2 - \lambda\bar{a} &= 0, \quad \bar{a}(T) = 0, \\ \bar{b}_s - \frac{1}{2\kappa_2}\bar{a}\bar{b} - 2\alpha - \lambda\bar{b} &= 0, \quad \bar{b}(T) = 0, \\ \bar{c}_s - \frac{1}{4\kappa_2}\bar{b}^2 + \alpha^2 - \lambda\frac{\bar{b}^2}{2\bar{a}} &= 0, \quad \bar{c}(T) = 0. \end{cases}$$

The closed form solution to this system of ODEs is given by

$$\bar{a}(s) = \kappa_2\left(-\lambda - \zeta + \frac{2\zeta}{1 - c_{\bar{a}}e^{\zeta s}}\right),$$

$$\bar{b}(s) = -\int_s^T 2\alpha(r)\exp\left(-\frac{1}{2\kappa_2}\int_s^r \bar{a}(w)dw - \lambda(r-s)\right)dr,$$

$$\bar{c}(s) = \int_s^T \left(-\frac{1}{4\kappa_2}\bar{b}^2(r) + \alpha^2(r) - \lambda\frac{\bar{b}^2(r)}{2\bar{a}(r)}\right)dr,$$

where the integration constant $c_{\bar{a}}$ is chosen such that the terminal conditions $\bar{a}(T) = 0$ is satisfied and we defined $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa_2}$. Given now the known form of $\overline{v}$, one may regard v as a second linear quadratic regulator problem with modified running costs, again this has a closed form solution

$$v(t, x) = \frac{1}{2}a(t)x^2 + b(t)x + c(t),$$

where now $a, b, c : [0, T] \mapsto \mathbb{R}$ solve a system of Riccati equations (which this time involve $\overline{a}, \overline{b}, \overline{c}, \beta$ and κ_1) and again may be solved explicitly.

$$v(t, x) = \frac{1}{2}a(t)x^2 + b(t)x + c(t),$$

where now $a, b, c : [0, T] \mapsto \mathbb{R}$ solve the following system of Riccati equations

$$\begin{cases} a_s - \frac{1}{2\beta(s)\kappa_1}a^2 + 2\beta(s) = 0, & a(T) = 0, \\ b_s - \frac{1}{2\beta(s)\kappa_1}ab - 2\beta(s)\alpha(s) = 0, & b(T) = 0, \\ c_s - \frac{1}{4\beta(s)\kappa_1}b^2 + \beta(s)\alpha(s)^2 + \beta(s)\lambda\left(-\frac{\bar{b}(s)^2}{2\bar{a}(s)} + \bar{c}(s)\right) = 0, & c(T) = 0. \end{cases}$$

This system admits the following solution

$$a(s) = \beta(s)\kappa_1\left(-\lambda - \zeta + \frac{2\zeta}{1 - c_a e^{\zeta s}}\right),$$

$$b(s) = -\int_s^T 2\alpha(r)2\beta(r)\exp\left(-\frac{1}{2\kappa_1}\int_s^r \frac{a(w)}{\beta(w)}dw\right)dr,$$

$$c(s) = \int_s^T \left(-\frac{1}{4\beta(r)\kappa_1}b(r)^2 + \beta(r)\alpha(r)^2 + \lambda\beta(r)\left(-\frac{\bar{b}(r)^2}{2\bar{a}(r)} + \bar{c}(r)\right)\right)dr,$$

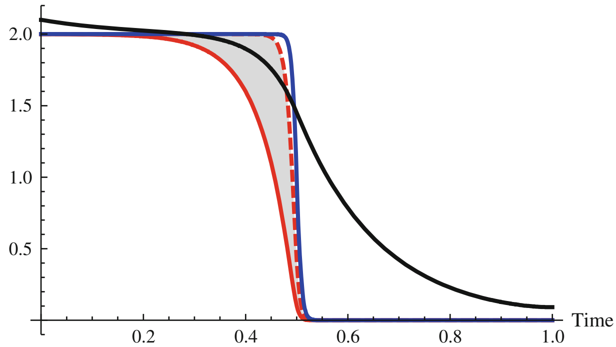

Fig. 2 In Example 8.2 before a jump of N the market orders are directed toward the *lowermost curve* and the passive orders toward the *dashed line*. The line cutting the shaded area between the previous two curves is the optimal stock holdings in the case when no jump occurs

where the integration constant c_a is chosen such that $a(T) = 0$ and we set $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa_1}$.

A calculation based upon the HJB PDE for v now gives that up until the random time τ, the optimal control is given $ds \times d\mathbb{P}$ a.e. on $[0, T] \times \Omega$ by

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-)) = -\dfrac{\overline{b}(s)}{\overline{a}(s)} - \hat{X}(s-), \\ \hat{u}_2(s, \hat{X}(s-)) = \dfrac{a(s)}{2\beta(s)\kappa_1}\left(-\dfrac{b(s)}{a(s)} - \hat{X}(s-)\right). \end{cases} \tag{8.1}$$

Exactly as before we see that the passive order is directed to the minimiser of $\overline{v}$ (with respect to x) whereas the market order towards the minimiser of v (with respect to x), again because the passive order foresees the jump in Z.

Within the framework of this example we can show that the hypothesised conjecture is false. For example Fig. 2 illustrates the situation with the monotonically decreasing target function $\alpha(s) = \tanh(100(0.5 - s)) + 1, T = 1, \lambda = 10$ and liquidity cost parameters $\kappa_1 = 10^{-2}$ and $\kappa_2 = 10^{-4}$. The stock holdings are assumed to be above the target function initially, i.e. $\hat{X}(0) = \alpha(0) + 0.1$.

Heuristically we start above a target which is decreasing so we would expect to only use market sell and passive sell orders, however this is not the case. The line cutting the shaded area indicates the path of the optimal trajectory up until the time of the first jump; it is decreasing showing that we use market orders as expected. However it enters the shaded region, where a passive order is placed which would take the stock holdings to the dashed line in the event of a jump and can be seen to be a passive buy order. Since paths like this occur with positive probability this demonstrates that even starting above a monotonically decreasing target function we cannot expect to use only sell orders. Similar examples can be constructed within this framework for other target functions.

We finish with a positive result, showing that for a constant target function orders always have the same sign so that the signal Z may then be interpreted as spread. This is because knowing a posteriori that orders have the same sign, we are only interested in (for example) the sell side of the order book and liquidity events on the buy side can be assumed independent of the spread on the sell side.

Lemma 8.3 *Suppose that* $\alpha(s,z)=c$ *for all* $(s,z)\in[t,T]\times\mathbb{R}^n$ *and some* $c\in\mathbb{R}$. *Then*

$$\mathrm{sign}(\hat{u}_1(s,\omega))=\mathrm{sign}(\hat{u}_2(s,\omega)),\quad ds\times d\mathbb{P}\ a.e.\ on\ [t,T]\times\Omega$$

Proof By translation we may assume $\alpha\equiv 0$. Suppose now that $\hat{X}(0)>0$, the case when $\hat{X}(0)<0$ is symmetric. Inspecting the expression for the performance functional, one can see that

$$J(t,0,z,u)>0=J(t,0,z,0)$$

for any control $u\in\mathcal{U}_t$, $u\not\equiv 0$ and $(t,z)\in[0,T]\times\mathbb{R}^n$. In particular we conclude that $v(t,0,z)=0$ so that thanks to the strict convexity and nonnegativity of v we see that $\tilde{\alpha}\equiv 0$. A consequence of this is that we can show (similarly to Proposition 6.2) that

$$\hat{u}_1(s,\omega)=-\hat{X}(s-,\omega),\quad ds\times d\mathbb{P}\text{ a.e. on }[t,T]\times\Omega.$$

Since for any $t\in[0,T]$ and control $u\in\mathcal{U}_t$ which is not identically zero on $[t,T]$ we have $J(t,0,z,u)>0$ it follows that once the optimal trajectory $\hat{X}$ hits zero, it stays there. Moreover as $\hat{X}$ evolves continuously before a jump we see that

$$\hat{X}(s)\geq 0\quad\text{for all } s\in[t,T]\text{ a.s.}$$

so that $\hat{u}_1$ is negative. Using the expression for P as a conditional expectation, see (6.2), we deduce that $P(s)\leq 0$ for all $s\in[t,T]$ a.s. From Theorem 3.2 we have the relation

$$\hat{u}_2(s,\omega)=\left[g_{u_2}(\cdot,Z(s,\omega))\right]^{-1}(P(s,\omega)),\quad ds\times d\mathbb{P}\text{ a.e. on }[t,T]\times\Omega,$$

showing that $\hat{u}_2$ is negative which then completes the proof. □

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Appendix A: Auxiliary results

We collect here two lemmata which are needed in the main body of the article.

Lemma A.1 *Given controls* $u,\bar{u}\in\mathcal{U}_t$ *such that* $X^u=X^{\bar{u}}ds\times d\mathbb{P}$ *a.e. on* $[t,T]\times\Omega$

(i) *The processes* X^u *and* $X^{\bar{u}}$ *are indistinguishable, i.e*

$$\sup_{s\in[t,T]}\left|X^u(s)-X^{\bar{u}}(s)\right|=0.$$

(ii) *The controls are identical, i.e*

$$u=\bar{u}\quad ds\times d\mathbb{P}\ a.e.\ on\ [t,T]\times\Omega.$$

Proof We write $X\triangleq X^u$ and $\bar{X}\triangleq X^{\bar{u}}$. From the $ds\times d\mathbb{P}$ equality we get

$$\mathbb{E}\left[\int_t^T\left(X(s)-\bar{X}(s)\right)^2ds\right]=0\tag{9.1}$$

which immediately implies

$$\int_t^T \left(X(s) - \bar{X}(s)\right)^2 ds = 0.$$

Since X and $\bar{X}$ are càdlàg semimartingales with finitely many jumps on $[t, T]$ we conclude that

$$\sup_{s\in[t,T)} \left|X(s) - \bar{X}(s)\right| = 0,$$

i.e equality on $[t, T)$. Since N is a Poisson process we have $\mathbb{P}(\Delta N(T) > 0) = 0$ so we can extend the equality to $[0, T]$ which establishes the first claim. A consequence of item (i) is that the quadratic variation is zero and thus we have

$$0 = \int_t^T d[X - \bar{X}, X - \bar{X}](s) = \int_t^T (u_1(s) - \bar{u}_1(s))^2 N(ds)$$

Taking expectation and using the L^2-property of the controls u and $\bar{u}$, we get

$$0 = \mathbb{E}\left[\int_t^T (u_1(s) - \bar{u}_1(s))^2 N(ds)\right] = \lambda\mathbb{E}\left[\int_t^T (u_1(s) - \bar{u}_1(s))^2 ds\right]$$

which implies $u_1 = \bar{u}_1$, $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$. Combining this with Eq. 9.1 and repeating the argument above leads to the following relation for the second components,

$$\int_t^T \left(\int_t^s u_2(r) - \bar{u}_2(r)dr\right)^2 ds = 0$$

which then implies that

$$\sup_{t\le s\le T} \left|\int_t^s u_2(r) - \bar{u}_2(r)dr\right| = 0.$$

Thus the total variation of the process $\int_t^{\cdot} u_2(s) - \bar{u}(s)ds$ is zero, which leads to

$$\int_t^T |u_2(s) - \bar{u}_2(s)|\, ds = 0$$

and so $u_2 = \bar{u}_2$, $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$, which completes the proof. □

Remark A.2 Observe that item (i) remains valid for any càdlàg process with finitely many jumps at exponentially distributed random times, in particular P, as in the second part of Proposition 6.5.

The next lemma provides a sufficient condition for the stochastic integral with respect to a compensated Poisson random measure to be a true martingale. We provide a proof for completeness.

Lemma A.3 *Let L be a Poisson random measure with compensator l and H be a predictable L-integrable process such that*

$$\mathbb{E}\left[\int_t^T \int_{\mathbb{R}^k} |H(s,\theta)| l(ds, d\theta)\right] < \infty.$$

Then the process

$$\int_t^{\cdot} \int_{\mathbb{R}^k} H(s,\theta)\tilde{L}(ds, d\theta)$$

is a true martingale, where $\tilde{L}$ denotes the compensated Poisson random measure.

Proof We proceed by an approximation argument. Let $n \in \mathbb{N}$ and define the truncated strategy $H^n(r,\theta) \triangleq H(r,\theta) \wedge n \vee (-n)$ and consider the process

$$M^n(s) \triangleq \int_t^s \int_{\mathbb{R}^k} H^n(r,\theta)\tilde{L}(dr, d\theta), \quad s \in [t, T].$$

The process M^n is a martingale thanks to [29, Theorem II.2.29]. We now have the following estimate,

$$\mathbb{E}\left[\sup_{t \le s \le T} \left|M^n(s) - M^m(s)\right|\right] \le 2\mathbb{E}\left[\int_t^T \int_{\mathbb{R}^k} \left|H^n(s,\theta) - H^m(s,\theta)\right| l(ds, d\theta)\right].$$

Letting m and n go to infinity and using the assumptions of the lemma, it follows then that $(M^n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{H}^1$, the Banach space of uniformly integrable martingales on $[t, T]$ equipped with the norm

$$\|M\|_{\mathcal{H}^1} \triangleq \mathbb{E}\left[\sup_{t \le s \le T} |M(s)|\right].$$

This sequence therefore has a limit M, which is also a martingale. On the other hand we deduce from the Dominated Convergence Theorem [29, Theorem IV.32] that

$$\lim_{n\to\infty} \sup_{t \le s \le T} \left| M^n(s) - \int_t^s \int_{\mathbb{R}^k} H(r,\theta)\tilde{L}(dr, d\theta)\right| = 0,$$

i.e. $(M^n)_{n\in\mathbb{N}}$ converges in UCP to

$$\int_t \int_{\mathbb{R}^k} H(s,\theta)\tilde{L}(ds, d\theta).$$

We thus conclude that this process is indistinguishable from M and hence a true martingale. □

Remark A.4 At first sight this lemma may appear obvious, however one must be careful as in general it is not true that the stochastic integral with respect to a compensated Poisson random measure is even a local martingale, see the example of Emery [17]. This motivates the need for the approximation in the above.

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